Chesapeake Section of the American Association of Physics Teachers

HIGHWAY MIRAGES

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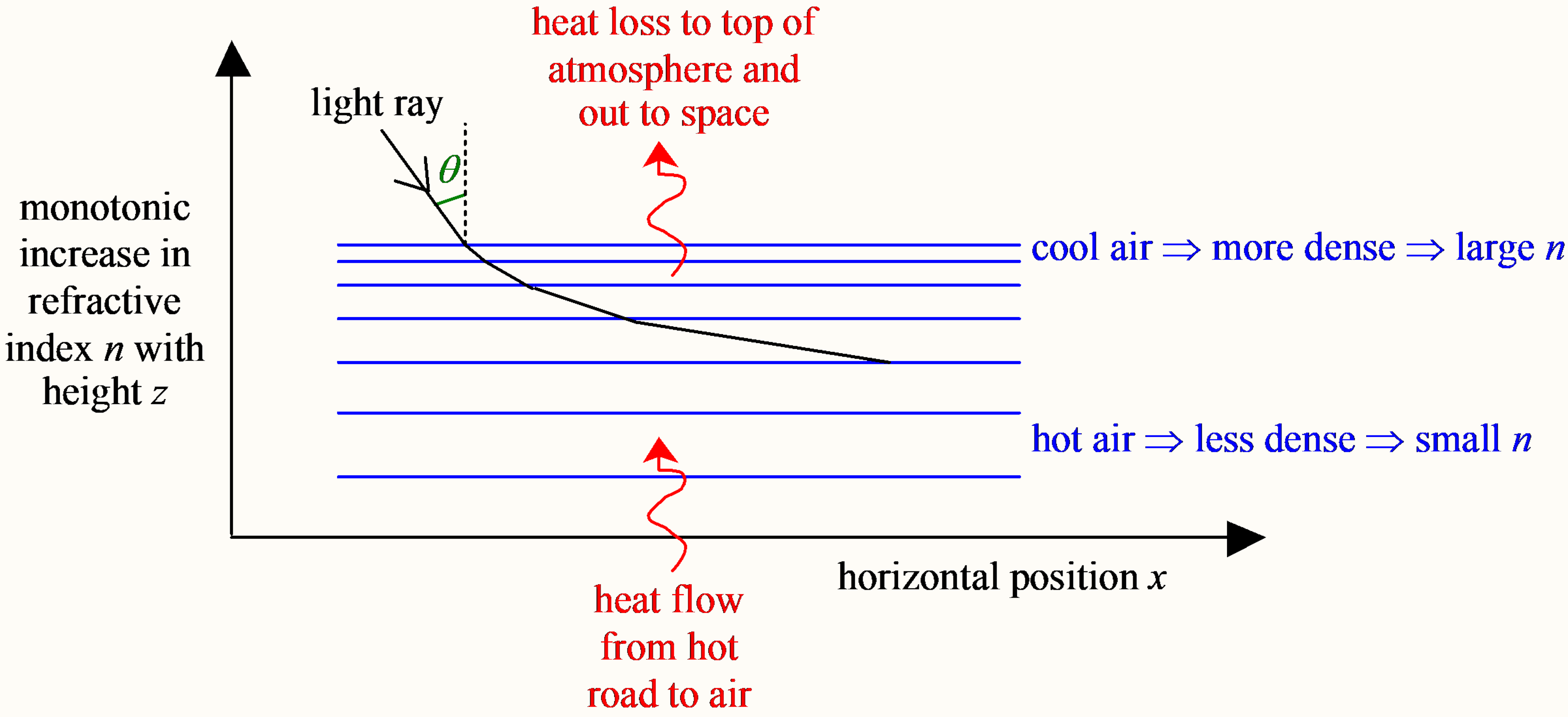
The air near a road heated by the sun can be modeled as a set of layers whose temperatures decrease with height. Consider a light ray incident from the sky at an acute angle θ.

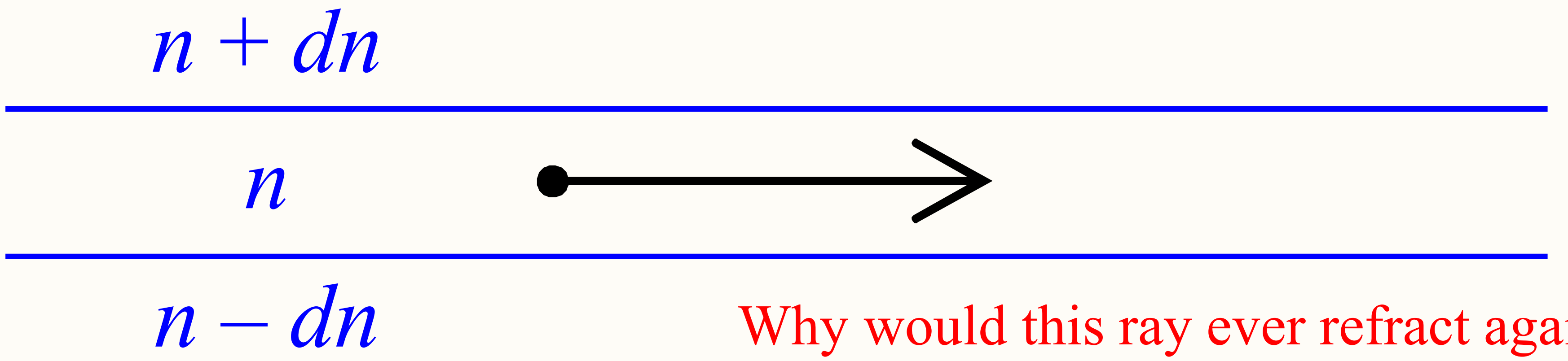


Will the ray get deflected ABOVE the horizontal BEFORE hitting the road?

Not unless θ starts out large enough. (There will be no deflection if $\theta = 0°$ which is a ray incident $\perp$ to the road.)

If θ starts out arbitrarily large, is there guaranteed to be a mirage if n monotonically increases with z?

A naïve view of Snell's law suggests that a mirage can NEVER happen! Because consider what happens when the ray reaches $\theta = 90°$:



Why would this ray ever refract again?

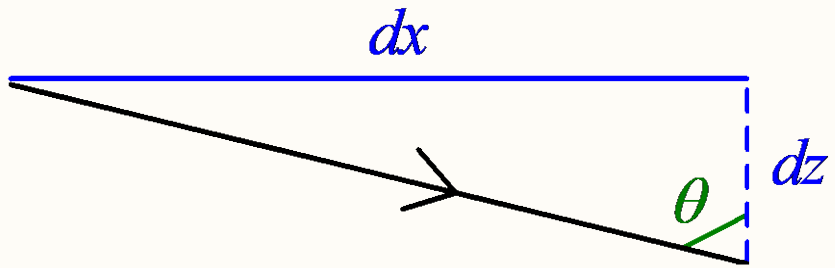
Mull this issue over while I show *mathematically* that we nevertheless *can* get a mirage using Snell's law alone.

Snell's law is $n \sin\theta = \text{constant}$.

Assume that a ray can reach $\theta = 90°$ (because the initial angle is large enough). At that point, the ray is at a height such that the index has the smallest value $n_{\min}$ that the ray will encounter on its path. We can then rewrite the constant above as

$$n \sin\theta = n_{\min} \sin 90° \quad \Rightarrow \quad \sin\theta = \frac{n_{\min}}{n}.$$

Now consider an infinitesimal portion of the ray's trajectory:



$$\sin\theta = \frac{dx}{\sqrt{dx^2 + dz^2}} = \frac{1}{\sqrt{1 + (dz/dx)^2}} = \frac{n_{\min}}{n} \quad \Rightarrow \quad \boxed{\frac{dx}{dz} = \frac{\pm 1}{\sqrt{(n/n_{\min})^2 - 1}}}$$

$$\text{where the sign is } \begin{cases} - & \text{for a downward traveling ray } (dz < 0) \\ + & \text{upward} \end{cases}$$

Given $n(z)$ the boxed DE can be solved for the trajectory $x(z)$.

Example 1: Linear variation in index with height

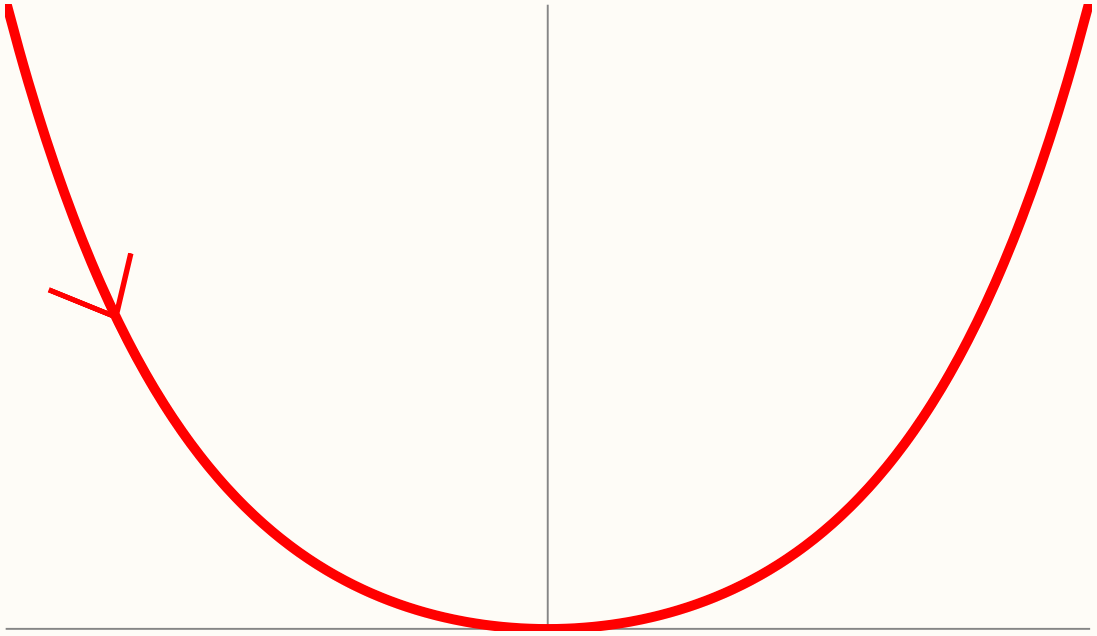
Let the refractive index be

$$n = n_{\min}(1 + kz)$$

such that $n = n_{\min}$ at $z = 0$. Here k is a parameter with units of reciprocal meters that is approximately proportional to the temperature gradient of the air. By choosing the constant of integration so that the ray reaches $z = 0$ at $x = 0$, the solution is

$$z = \frac{\cosh(kx) - 1}{k}$$

as can be checked by substitution into the DE.



(Another physics application of catenaries to add to your repertoire!)

Example 2: A more complicated variation in index with height

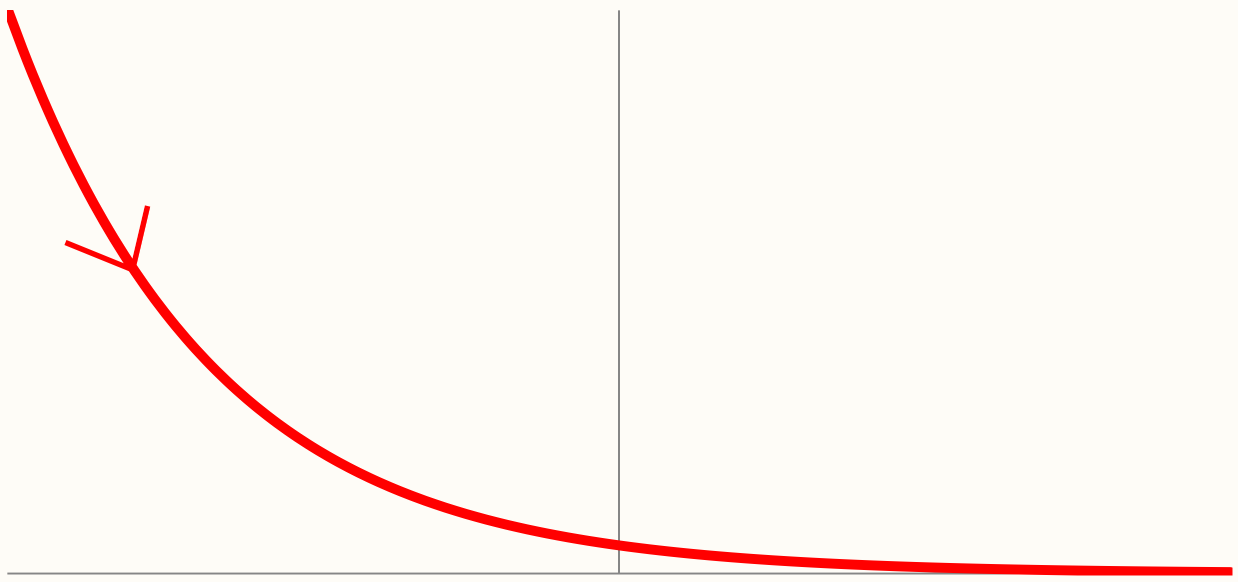
Suppose the index varies as

$$n = n_{\min}\sqrt{1+(kz)^2}$$

with the same comments about $n_{\min}$ and k as in the first example. Direct integration leads to

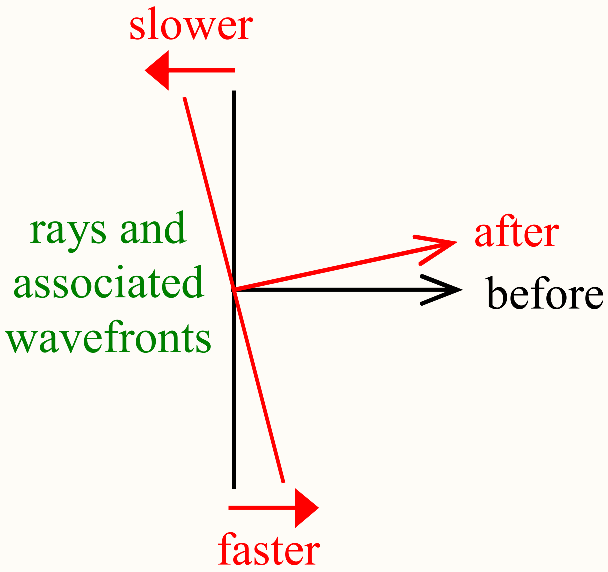
$$z = z_0 e^{-kx}$$

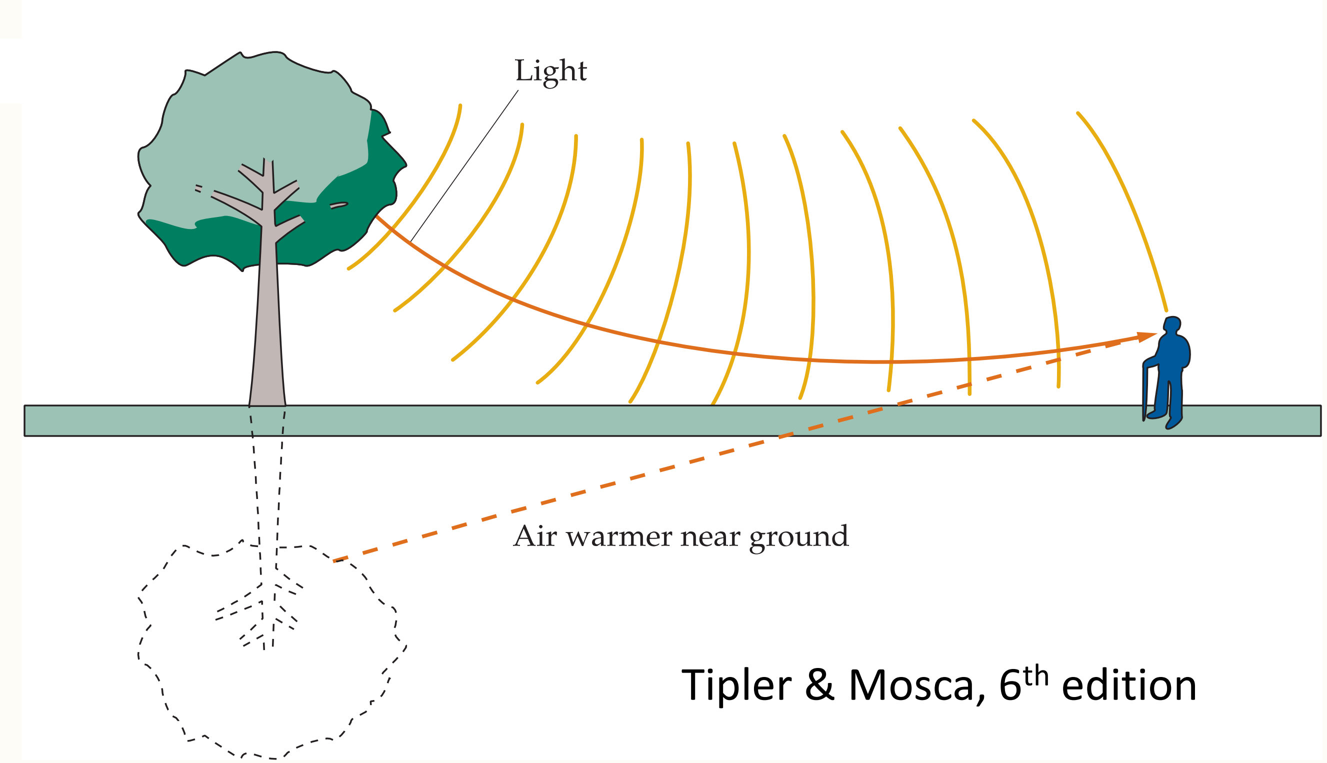
where z_0 is the height of the ray at $x = 0$. This time the ray does NOT turn around vertically!



So how *does* a ray turn around?

The most common textbook explanation goes as follows:

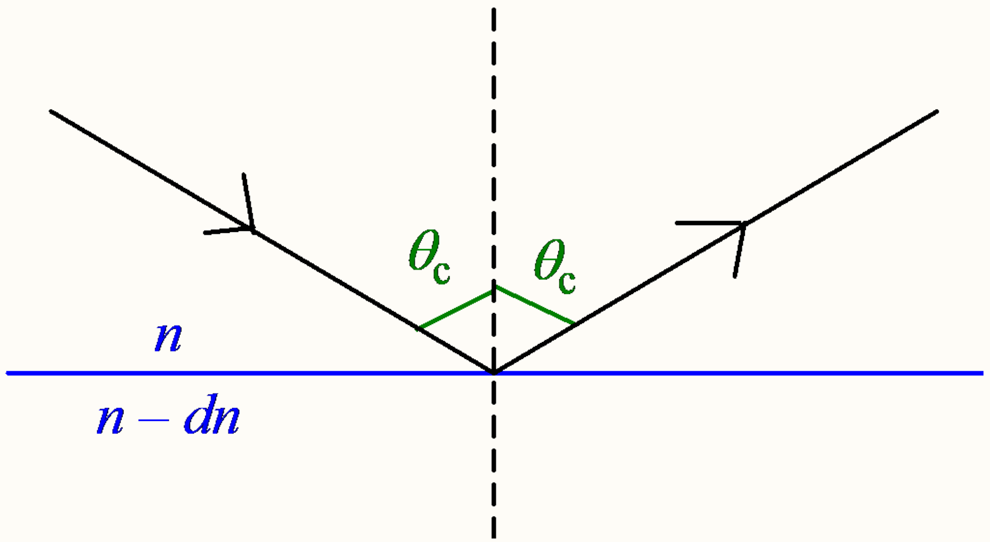




Tipler & Mosca, 6th edition

That explanation requires the wavefronts to have a finite vertical cross section.
But our analysis above used rays of *zero* thickness.
How can we explain the turnaround using only a ray model?

<u>ANSWER</u>: A ray reflects when it reaches the critical angle.



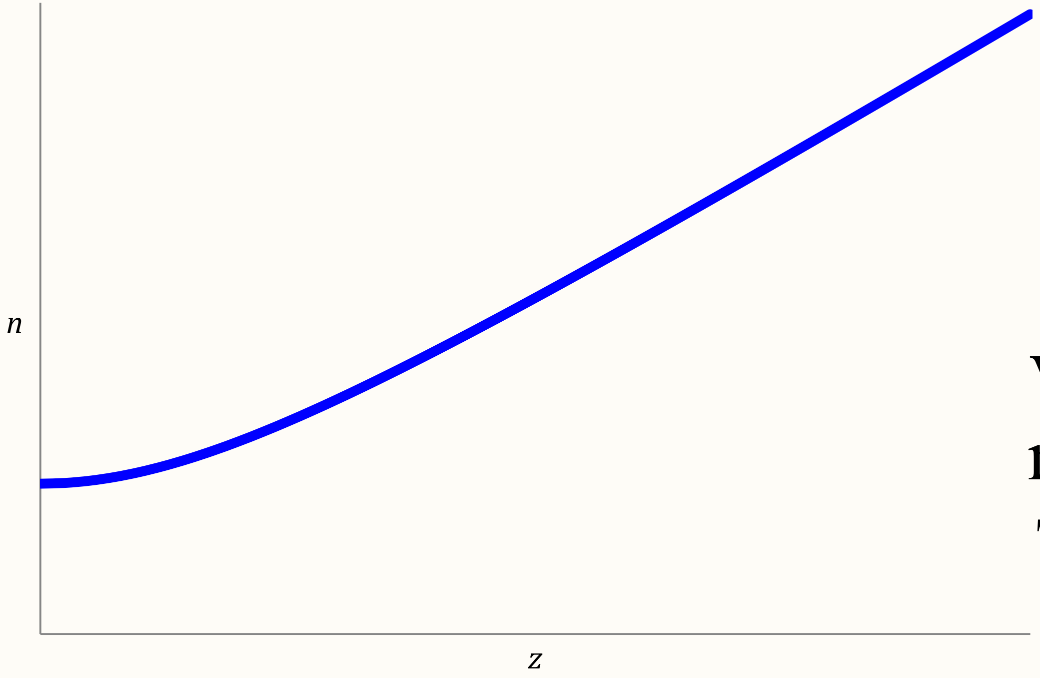
Here $\sin\theta_c = \dfrac{n - dn}{n} \to 1$ so that $\theta_c \to 90°$.

That is, a ray will *totally reflect* once the ray gets infinitesimally close to 90°.

To put it simply, the ray reflects "at" 90°.

(This result is consistent with the fact that *any* surface becomes 100% reflecting at grazing incidence.)

Okay… but why did that NOT happen in example 2?



Because n becomes constant as $z \longrightarrow 0$ for this particular example. But if n stops changing from one vertical layer to the next, then there is no longer any refraction or reflection! The ray gets "stuck" at the 90° angle.

Many thanks to Prof. Craig Wiegert, Assoc. Dept. Chair at the University of Georgia for helpful discussions.

QUESTIONS?

Comments also welcome by email to mungan@usna.edu.