

Catenaries and Suspension Bridges – The Shape of a Hanging String/Cable

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FIG. 1. The Clifton Suspension Bridge in Bristol, UK. Opened in 1864. Photo by Gothick, Creative Commons Attribution-Share Alike 3.0 Unported license.

I. INTRODUCTION

We see suspension bridges all around the world and we all recognize the distinctive shape of the cables that support them. The question is, what is that shape? In these notes, I will derive what that shape is and introduce a simple demo that can be performed in class.

I begin by discussing the shape of a catenary, namely, the shape of a hanging string/cable which is supporting its own weight. I will first use the variational method to derive the shape of the catenary, and then present a non-variational method which, naturally, leads to the same result.

I will then apply the second method to the problem of suspension bridges and derive the shape of the suspension-bridge cable which is supporting the weight of the bridge hanging from it.

Finally, I will discuss a simple demonstration that can be performed in class which compares the two results.

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II. REVIEW OF THE VARIATIONAL METHOD

The variational method is used to derive the Euler-Lagrange equations of motion from the action S , via the application of the principle of least action $\delta S = 0$:

$$S = \int_{t_A}^{t_B} dt L(y, \dot{y}), \quad \dot{y} = \frac{dy}{dt}, \quad \xrightarrow{\delta S=0} \quad \frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0. \quad (1)$$

However, it can be used in many other contexts as well. Whenever you need to minimize (or maximize) an integral of a functional $F(y, y')$ of a function y and its derivative y' , you obtain the Euler-Lagrange equation. To make sure everyone is on board, I begin with a brief review of the formalism.

A. Minimization Problem

We would like to find the function $y(x)$ which minimizes an integral of the form

$$P = \int_a^b dx F(y, y'), \quad y' = \frac{dy}{dx}, \quad (2)$$

under the condition that $y(a)$ and $y(b)$ are fixed. If we vary the function $y(x)$ by a variation $\delta y(x)$,

$$y(x) \rightarrow y(x) + \delta y(x), \quad \delta y(a) = \delta y(b) = 0, \quad (3)$$

then the change in P will be

$$\begin{aligned} \delta P &= \int_a^b dx F(y + \delta y, y' + \delta y') \\ &= \int_a^b dx \left\{ \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right\} \\ &= \int_a^b dx \left[\frac{\partial F}{\partial y} \delta y + \left\{ \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \delta y \right) - \left(\frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y \right\} \right] \\ &= \int_a^b dx \left\{ \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y + \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \delta y \right) \right\} \\ &= \int_a^b dx \left\{ \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y \right\} + \int_a^b dx \left\{ \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \delta y \right) \right\} \\ &= \int_a^b dx \left\{ \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y \right\} + \underbrace{\left[\frac{\partial F}{\partial y'} \delta y \right]_a^b}_{=0} \\ &= \int_a^b dx \left\{ \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y \right\}. \end{aligned} \quad (4)$$

In the penultimate line we have used $\delta y(a) = \delta y(b) = 0$. Therefore, for $\delta P = 0$ regardless of the choice of δy , we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0. \quad (5)$$

This is the Euler-Lagrange equation.

B. Minimization with a Global Constraint

Now, let's say that we want to minimize some integral

$$P = \int_a^b dx F(y, y'), \quad (6)$$

under the constraint that some other integral is constant:

$$Q = \int_a^b dx G(y, y') = \text{constant}. \quad (7)$$

We need to find the condition that $y(x)$ must satisfy so that the variation of P is zero,

$$\delta P = \int_a^b dx \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y(x) = 0, \quad (8)$$

with the restriction that $\delta y(x)$ satisfies the condition

$$\delta Q = \int_a^b dx \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \frac{\partial G}{\partial y'} \right) \delta y(x) = 0, \quad (9)$$

so that the value of Q does not change.

This is actually much easier than it may seem. All we need is to notice that the integrals in Eqs. (8) and (9) can be considered inner products of two infinite dimensional vectors whose components are labelled by the continuous parameter x . Let

$$f(x) \equiv \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right), \quad g(x) \equiv \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \frac{\partial G}{\partial y'} \right), \quad h(x) \equiv \delta y(x). \quad (10)$$

Then, we can write

$$\delta P = \int_a^b dx f(x)h(x), \quad \delta Q = \int_a^b dx g(x)h(x). \quad (11)$$

This is no different than writing

$$\delta P = \sum_{i=1}^N f_i h_i, \quad \delta Q = \sum_{i=1}^N g_i h_i, \quad (12)$$

for finite dimensional vectors. So Eqs. (8) and (9) can be interpreted to mean: find the condition that vector $f(x)$ must satisfy so that it is perpendicular to any vector $h(x)$ which is perpendicular to vector $g(x)$. The answer is that vector $f(x)$ must be parallel to vector $g(x)$, *i.e.*

$$f(x) + \lambda g(x) = 0, \quad (13)$$

for some constant λ . In its original form, we get

$$\left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) + \lambda \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \frac{\partial G}{\partial y'} \right) = 0. \quad (14)$$

But this is just the Euler-Lagrange equation we would get if we were asked to minimize the integral

$$\int_a^b dx (F + \lambda G). \quad (15)$$

The constant λ is called the *Lagrange multiplier*.

III. SHAPE OF STRING SUSPENDED BETWEEN TWO POINTS

A. The Problem

Let's figure out what the shape of a string of uniform linear mass density ρ will be when it is suspended between two points A and B . The string will settle into a shape that minimizes the gravitational potential energy.

Take the x -axis in the horizontal direction and the y -axis in the vertical direction (up). If we describe the shape of the string by specifying y as a function of x , $y(x)$, then the potential energy of the string between points $(x, y(x))$ and $(x + dx, y(x + dx) = y + dy)$ will be

$$\rho g y \sqrt{dx^2 + dy^2} = \rho g y \sqrt{1 + (y')^2} dx, \quad (16)$$

where

$$y' \equiv \frac{dy}{dx}. \quad (17)$$

So the total potential energy will be

$$V = \rho g \int_a^b y \sqrt{1 + (y')^2} dx, \quad (18)$$

where a and b are the x -coordinates of the points A and B , respectively. We need to minimize this under the constraint that the total length of the string does not change. The length of the string is given by the integral

$$\ell = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + (y')^2} dx. \quad (19)$$

This can be done using the Lagrange multiplier method discussed above.

B. The Solution

So to solve the string suspension problem, all we need to do is apply the Euler-Lagrange equation to the function

$$F = y \sqrt{1 + (y')^2} + \lambda \sqrt{1 + (y')^2} = (y + \lambda) \sqrt{1 + (y')^2}. \quad (20)$$

(We have factored out the ρg for simplicity.) Since

$$\begin{aligned} \frac{\partial F}{\partial y} &= \sqrt{1 + (y')^2}, & \frac{\partial F}{\partial y'} &= (y + \lambda) \frac{y'}{\sqrt{1 + (y')^2}}, \\ \frac{d}{dx} \frac{\partial F}{\partial y'} &= \frac{(y')^2}{\sqrt{1 + (y')^2}} + (y + \lambda) \left(\frac{y''}{\sqrt{1 + (y')^2}} - \frac{(y')^2 y''}{\sqrt{(1 + (y')^2)^3}} \right) \\ &= \frac{(y')^2}{\sqrt{1 + (y')^2}} + (y + \lambda) \frac{y''}{\sqrt{1 + (y')^2}} \left(1 - \frac{(y')^2}{1 + (y')^2} \right) \\ &= \frac{(y')^2}{\sqrt{1 + (y')^2}} + (y + \lambda) \frac{y''}{\sqrt{(1 + (y')^2)^3}}, \end{aligned} \quad (21)$$

we find

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x} - \frac{d}{dx} \frac{\partial F}{\partial y'} = \sqrt{1 + (y')^2} - \frac{(y')^2}{\sqrt{1 + (y')^2}} - (y + \lambda) \frac{y''}{\sqrt{(1 + (y')^2)^3}} \\ &= \frac{1 + (y')^2}{\sqrt{1 + (y')^2}} - \frac{(y')^2}{\sqrt{1 + (y')^2}} - (y + \lambda) \frac{y''}{\sqrt{(1 + (y')^2)^3}} \\ &= \frac{1}{\sqrt{1 + (y')^2}} - (y + \lambda) \frac{y''}{\sqrt{(1 + (y')^2)^3}} = \frac{1}{\sqrt{1 + (y')^2}} \left\{ 1 - (y + \lambda) \frac{y''}{1 + (y')^2} \right\}, \end{aligned} \quad (22)$$

or

$$0 = 1 - (y + \lambda) \frac{y''}{1 + (y')^2}. \quad (23)$$

Divide by $(y + \lambda)$ and multiply by $2y'$, then integrate:

$$0 = \frac{2y'}{y + \lambda} - \frac{2y'y''}{1 + (y')^2} .$$

$$\downarrow$$

$$\text{constant} = 2 \ln(y + \lambda) - \ln(1 + (y')^2) = \ln \frac{(y + \lambda)^2}{1 + (y')^2} . \quad (24)$$

Therefore,

$$\frac{(y + \lambda)^2}{1 + (y')^2} = \text{constant} \equiv \frac{1}{k^2} . \quad (25)$$

Solve for y' :

$$y' = \frac{dy}{dx} = \sqrt{k^2(y + \lambda)^2 - 1}$$

$$\downarrow$$

$$k dx = \frac{k dy}{\sqrt{k^2(y + \lambda)^2 - 1}} . \quad (26)$$

We can drop the λ here by shifting the origin in the y direction. We get

$$k dx = \frac{k dy}{\sqrt{k^2 y^2 - 1}} . \quad (27)$$

Change variable to $\xi = ky$. Then

$$k dx = \frac{d\xi}{\sqrt{\xi^2 - 1}} , \quad (28)$$

which can be integrated to yield

$$kx = \cosh^{-1} \xi = \cosh^{-1} ky , \quad (29)$$

or

$$y = \frac{1}{k} \cosh kx . \quad (30)$$

This curve is known as the *catenary*.

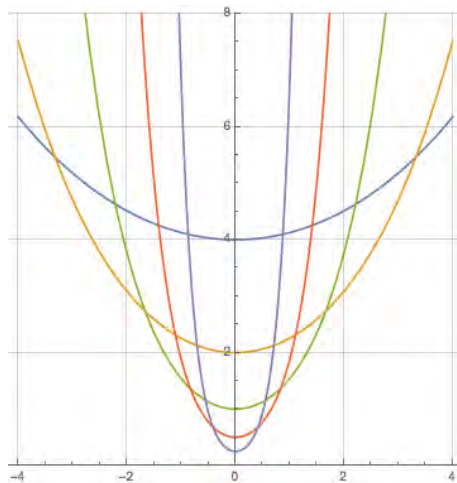


FIG. 2. The catenary. $y(x) = \cosh(kx)/k$ is shown for $k = 1/4$ (blue), $1/2$ (yellow), 1 (green), 2 (red), and 4 (purple).

The parameter k is determined by the positions of the points A and B and the length of the string ℓ . For the sake of simplicity, let's consider the case in which points A and B are at the same height and their horizontal separation is 2Δ . We take the x -coordinate of point A to be $-\Delta$, and that of point B to be Δ . The length of the string between A and B will be given by

$$\ell = \int_{-\Delta}^{\Delta} dx \sqrt{1 + (y')^2} . \quad (31)$$

Since

$$\begin{aligned} y(x) &= \frac{1}{k} \cosh(kx) \\ &\downarrow \\ y'(x) &= \sinh(kx) , \end{aligned} \quad (32)$$

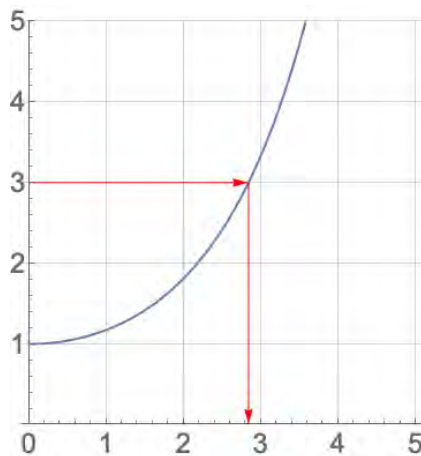
we find

$$\begin{aligned} \ell &= \int_{-\Delta}^{\Delta} dx \sqrt{1 + \sinh^2(kx)} = \int_{-\Delta}^{\Delta} dx \sqrt{\cosh^2(kx)} = \int_{-\Delta}^{\Delta} dx \cosh(kx) \\ &= \left[\frac{1}{k} \sinh(kx) \right]_{-\Delta}^{\Delta} = \frac{1}{k} \{ \sinh(k\Delta) - \sinh(-k\Delta) \} = \frac{2}{k} \sinh(k\Delta) , \end{aligned} \quad (33)$$

or

$$\frac{\ell}{2\Delta} = \frac{\sinh(k\Delta)}{k\Delta} \equiv f(k\Delta) . \quad (34)$$

Now, I do not know how to solve this for k analytically, but this can be solved graphically. The graph for the function $f(\eta) = \sinh(\eta)/\eta$ looks like



so you can read off the value of $k\Delta$ corresponding to a given ratio $\ell/(2\Delta)$ and from that determine k . (Or you can use Mathematica to solve the equation numerically.) Note that there is no solution if $\ell < 2\Delta$, which of course means that you cannot span two points separated by 2Δ with a string shorter than that.

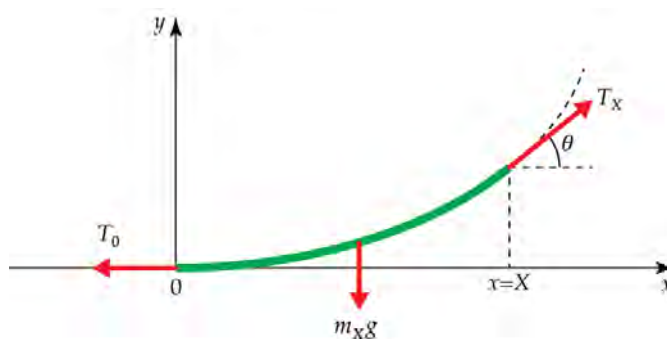


FIG. 3. Forces on a section of the suspended string.

IV. NON-VARIATIONAL APPROACH

A. The Catenary

The same result can be obtained via a non-variational method as follows. Choose your coordinate system so that $(x, y) = (0, 0)$ is the lowest point of your suspended string. If the shape of the string is given by the function $y(x)$, then $y(0) = 0$ and $y'(0) = 0$. Now, consider the forces acting on the part of the string between $x = 0$ and $x = X$. The length of the string between these points is

$$\ell_X = \ell(0 < x < X) = \int_0^X \sqrt{1 + (y')^2} dx, \quad (35)$$

so the mass of that section is

$$m_X = \rho \ell_X = \rho \int_0^X \sqrt{1 + (y')^2} dx, \quad (36)$$

and the gravity acting on it

$$m_X g = \rho g \ell_X = \rho g \int_0^X \sqrt{1 + (y')^2} dx. \quad (37)$$

Denote the magnitude of the tension on the string at $x = 0$ by T_0 , and that at $x = X$ by T_X . T_0 is pointing horizontally left, while T_X is pointing at an angle θ from the horizontal where $\tan \theta = y'(X)$. Since the three forces acting on the section under consideration must cancel, we must have

$$T_0 = T_X \cos \theta, \quad m_X g = T_X \sin \theta, \quad (38)$$

that is

$$y'(X) = \frac{T_X \sin \theta}{T_X \cos \theta} = \frac{m_X g}{T_0} = \frac{\rho g}{T_0} \int_0^X \sqrt{1 + (y')^2} dx. \quad (39)$$

Taking the derivative of both sides with respect to X , we obtain

$$y''(X) = \frac{\rho g}{T_0} \sqrt{1 + \{y'(X)\}^2}, \quad (40)$$

or

$$\frac{y''}{\sqrt{1 + (y')^2}} = \frac{\rho g}{T_0} \equiv k. \quad (41)$$

We suppress the arguments of the functions to simply the notation from this point on. Multiply both sides by y' to obtain

$$\frac{y' y''}{\sqrt{1 + (y')^2}} = k y'. \quad (42)$$

Integrate:

$$\sqrt{1 + (y')^2} = ky + 1, \quad (43)$$

where we have chosen the integration constant so that $y(0) = 0$ implies $y'(0) = 0$. Solve for y' and we find

$$y' = \sqrt{(ky + 1)^2 - 1}, \quad (44)$$

or

$$\frac{y'}{\sqrt{(ky + 1)^2 - 1}} = 1, \quad (45)$$

that is

$$\frac{kdy}{\sqrt{(ky + 1)^2 - 1}} = kdx. \quad (46)$$

Change variable to $\xi = (ky + 1)$ to obtain

$$\frac{d\xi}{\sqrt{\xi^2 - 1}} = kdx. \quad (47)$$

Solve:

$$\cosh^{-1}(\xi) = kx, \quad (48)$$

that is:

$$\begin{aligned} \xi &= ky + 1 = \cosh(kx) \\ &\downarrow \\ y &= \frac{1}{k} [\cosh(kx) - 1], \end{aligned} \quad (49)$$

and we see that we recover the same result as before. (The -1 inside the square brackets simply shifts the graph downwards so that $y(0) = 0$.)

B. Suspension Bridge

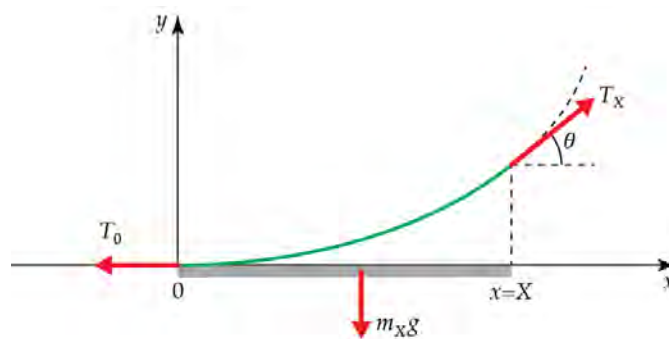


FIG. 4. Forces on a section of a suspension bridge. We assume that the mass of the cable is negligible compared to that of the suspended bridge section.

When a bridge is suspended from the cable (string) the shape will be slightly different. If we assume that the mass of the cable is negligible compared to the mass of the bridge, and that the bridge has linear mass density μ , then the mass supported by the cable between $x = 0$ and $x = X$ will be given simply by

$$m_X = \mu X, \quad (50)$$

and the gravity acting on it

$$m_X g = \mu g X . \quad (51)$$

Eq. (39) will be modified to

$$y'(X) = \frac{T_X \sin \theta}{T_X \cos \theta} = \frac{m_X g}{T_0} = \frac{\mu g}{T_0} X . \quad (52)$$

This can be trivially integrated to yield

$$y(X) = \frac{\mu g}{2T_0} X^2 , \quad (53)$$

or setting $\mu g/T_0 \equiv K$, and rewriting X as x , we obtain

$$y = \frac{K}{2} x^2 . \quad (54)$$

The constant K must again be determined from the condition that the length of the string is fixed. This time,

$$y' = Kx , \quad (55)$$

and we have

$$\ell = \int_{-\Delta}^{\Delta} dx \sqrt{1 + (y')^2} = \int_{-\Delta}^{\Delta} dx \sqrt{1 + (Kx)^2} , \quad (56)$$

or

$$K\ell = \int_{-K\Delta}^{K\Delta} d\xi \sqrt{1 + \xi^2} , \quad \xi \equiv Kx . \quad (57)$$

Change variable: $\xi = \sinh(\zeta)$, $d\xi = d\zeta \cosh(\zeta)$. Then

$$\begin{aligned} K\ell &= \int_{-\sinh^{-1}(K\Delta)}^{\sinh^{-1}(K\Delta)} d\zeta \cosh^2 \zeta = \int_{-\sinh^{-1}(K\Delta)}^{\sinh^{-1}(K\Delta)} d\zeta \frac{1 + \cosh 2\zeta}{2} \\ &= \left[\frac{\zeta}{2} + \frac{\sinh 2\zeta}{4} \right]_{-\sinh^{-1}(K\Delta)}^{\sinh^{-1}(K\Delta)} = \sinh^{-1}(K\Delta) + \left[\frac{\sinh \zeta \cosh \zeta}{2} \right]_{-\sinh^{-1}(K\Delta)}^{\sinh^{-1}(K\Delta)} \\ &= \sinh^{-1}(K\Delta) + K\Delta \sqrt{1 + (K\Delta)^2} , \end{aligned} \quad (58)$$

or

$$\frac{\ell}{2\Delta} = \frac{\sinh^{-1}(K\Delta)}{2K\Delta} + \frac{1}{2} \sqrt{1 + (K\Delta)^2} \equiv g(K\Delta) . \quad (59)$$

Again, I do not know how to write Δ as an analytic function of $\ell/(2\Delta)$ so we use a graphical method (or Mathematica). The graph of $g(\eta) = \sinh^{-1}(\eta)/(2\eta) + \sqrt{1 + \eta^2}/2$ looks like:



from which we can read off the value of $K\Delta$ when given $\ell/(2\Delta)$, from which we can determine K .

C. Catenary \leftrightarrow Suspension Bridge Comparison

The difference in the shapes of catenaries and suspension-bridge-cables (parabolas) are shown below in FIG. 5. The seven cases shown are for

$$\frac{\ell}{2\Delta} = 1.1, 1.2, 1.3, 1.4, 1.5, 2.0, 3.0. \quad (60)$$

As is evident from the graph, the difference is hardly noticeable for small $\ell/(2\Delta)$. The shapes of actual suspension-bridge cables fall somewhere in between the two cases since they support both their own weight and the weight of the bridge. The small difference is important for the engineers who are constructing the bridge, but it is impossible to tell the difference from photographs taken from afar.

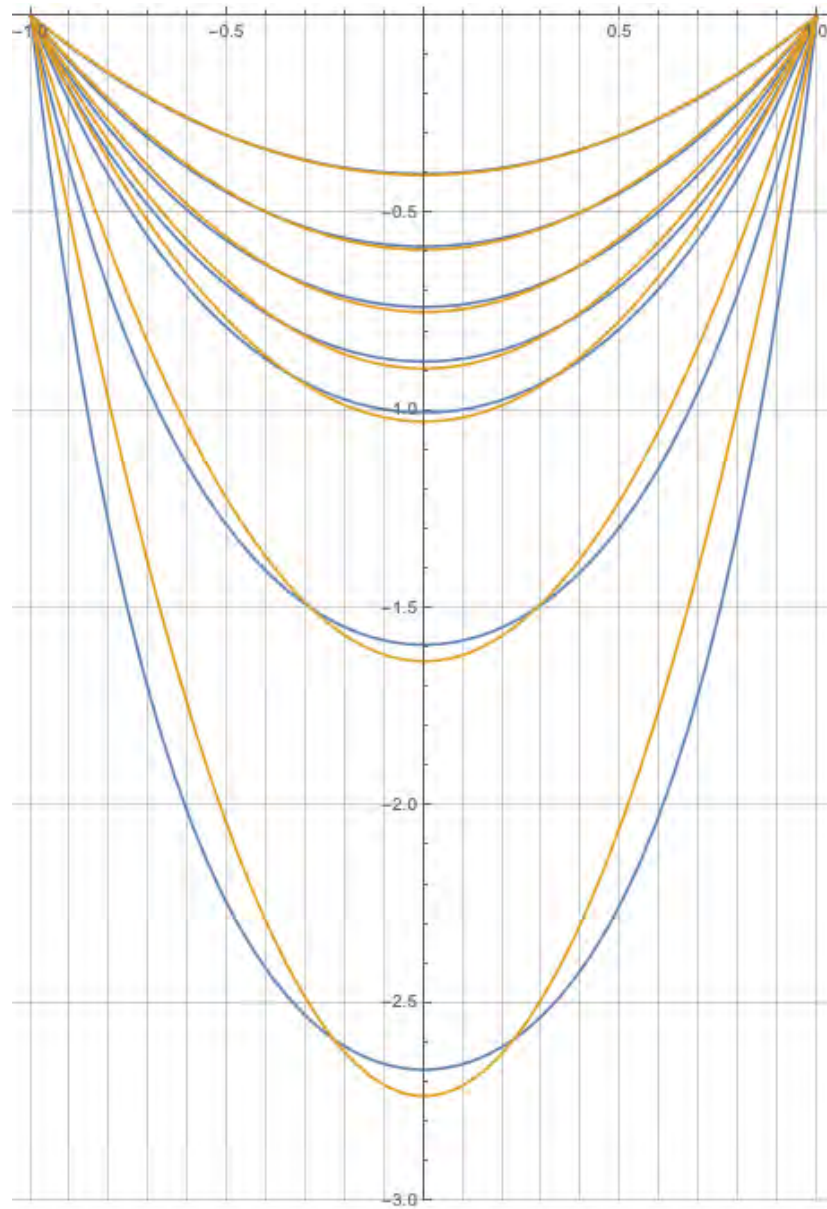


FIG. 5. Comparison of the shapes of catenaries (blue) and suspension-bridge-cables (parabolas, yellow). The cases shown are from top to bottom $\ell/(2\Delta) = 1.1, 1.2, 1.3, 1.4, 1.5, 2,$ and 3 .

V. THE DEMOS

A. Catenary

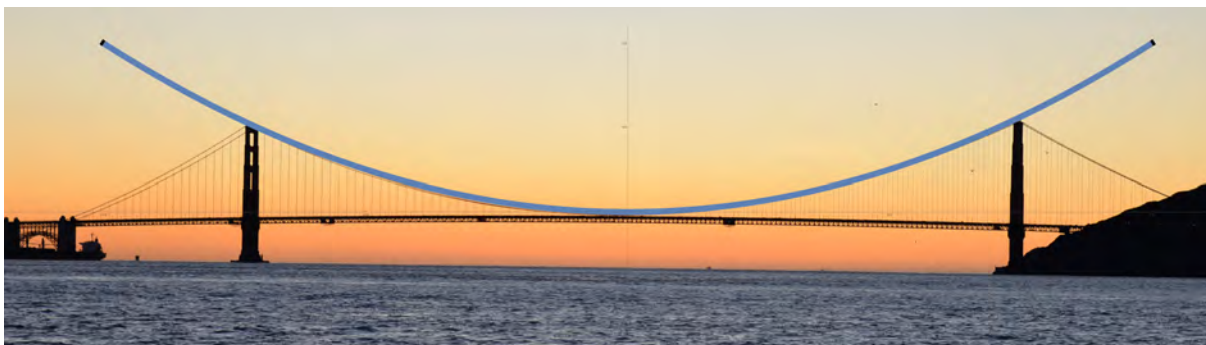
Plot a graph of the hyperbolic cosine function using Mathematica (or Desmos, or some other program) and project it onto a screen. Hold a piece of string against it and demonstrate that it does indeed fall into the shape of a catenary.

B. Suspension Bridge

Use Mathematica to overlay a graph of a parabola onto a photo of a suspension bridge. You can "Insert" the photo into Mathematica as a "Picture" and then use the "Overlay" command to superpose a graph onto it.



Graph overlaid over photo by Gothick, Creative Commons Attribution-Share Alike 3.0 Unported license.



Graph overlaid over photo by:

Author: D Ramey Logan

Golden Gate Bridge Dec 15 2015 by D Ramey Logan.jpg from Wikimedia Commons

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C. Catenary and Suspension Bridge Comparison

To be able to tell the difference between a catenary and a parabola, we need $\ell/(2\Delta)$ to be large. So let's use $\ell/(2\Delta) = 3$. We will prepare a piece of string 6 feet long and suspend it between two points that are 2 feet apart. If we attach weights to it so that they are equally spaced horizontally when the string is in the shape of a parabola, we can simulate a horizontal bridge of uniform mass density hanging from it.

Things to prepare

1. 6'-and-a-few-inches length piece of light string $\times 2$.
2. 19 fishing weights.

On one of the strings, attach the fishing weights at the following locations from the center:

1. 0 inches (center)
2. 1.26 inches (3.2 cm, both sides)
3. 2.82 inches (7.2 cm, both sides)
4. 4.85 inches (12.3 cm, both sides)
5. 7.45 inches (18.9 cm, both sides)
6. 10.6 inches (27.0 cm, both sides)
7. 14.4 inches (36.7 cm, both sides)
8. 18.9 inches (47.9 cm, both sides)
9. 23.9 inches (60.8 cm, both sides)
10. 29.6 inches (75.3 cm, both sides)
11. 36.0 inches (91.4 cm. No weights here. Just mark the spots since this is where you need to hold the string.)

Project FIG. 5 onto a screen and adjust the magnification so that the points where the strings are suspended from are separated by two feet. Hold the strings with and without the weights against the screen and compare the shapes with the graphs.

How to determine where to place the weights:

For different values of ℓ and Δ , the locations of the weights can be determined using the function $g(\eta)$ we defined in Eq. (59), namely:

$$g(\eta) = \frac{\sinh^{-1}(\eta)}{2\eta} + \frac{\sqrt{1+\eta^2}}{2}. \quad (61)$$

If we want to simulate the vertical cables of the suspension bridge being at

$$x_n = \frac{n}{N}\Delta, \quad n = 0, 1, 2, \dots, N-1, \quad (62)$$

we need to attach the weights at positions

$$\ell_n = \Delta g(x_n), \quad n = 0, 1, 2, \dots, N-1 \quad (63)$$

from the center.