Optimal launch angles:

Novel perspectives of an ancient problem

R.K.P. Zia

Center for Soft Matter and Biological Physics,
Physics Department, Virginia Tech
and
Department of Physics, University of Houston

Michael J. Ruiz

Department of Physics & Astronomy,
University of North Carolina at Asheville

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Revisiting the optimal shot-put release angle:

solving a maximization problem **with geometry**

R.K.P. Zia

Center for Soft Matter and Biological Physics,
Physics Department, Virginia Tech
and
Department of Physics, University of Houston

Michael J. Ruiz

Department of Physics & Astronomy,
University of North Carolina at Asheville

CSAAPT
April 2nd, 2022

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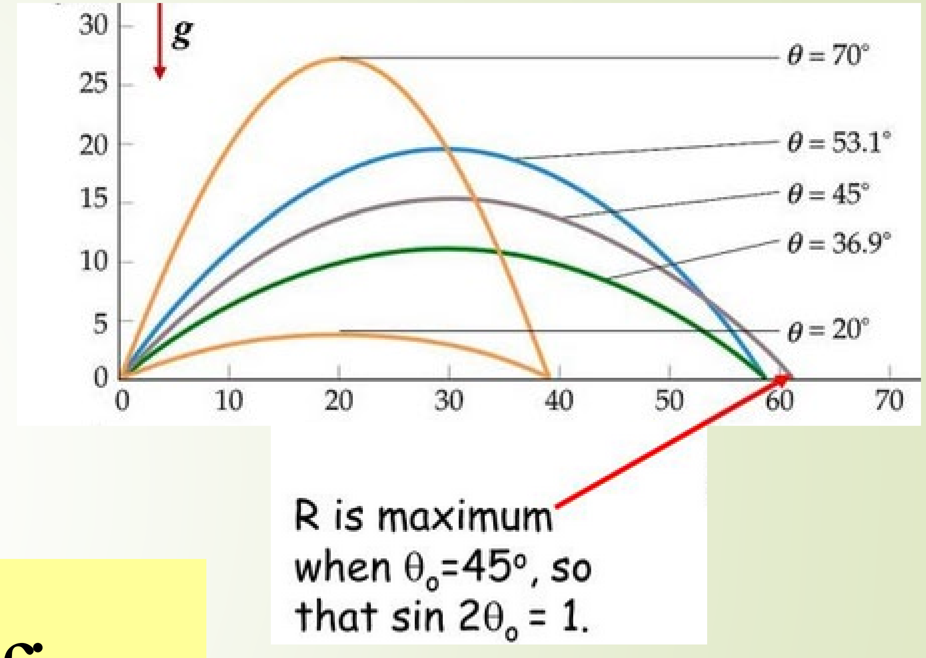
Ancient knowledge:



The "optimal angle" to fire a canon ball is 45° !

…to achieve the maximum range on flat ground

https://image3.slideserve.com/6601227/horizontal-range-of-a-projectile-l.jpg

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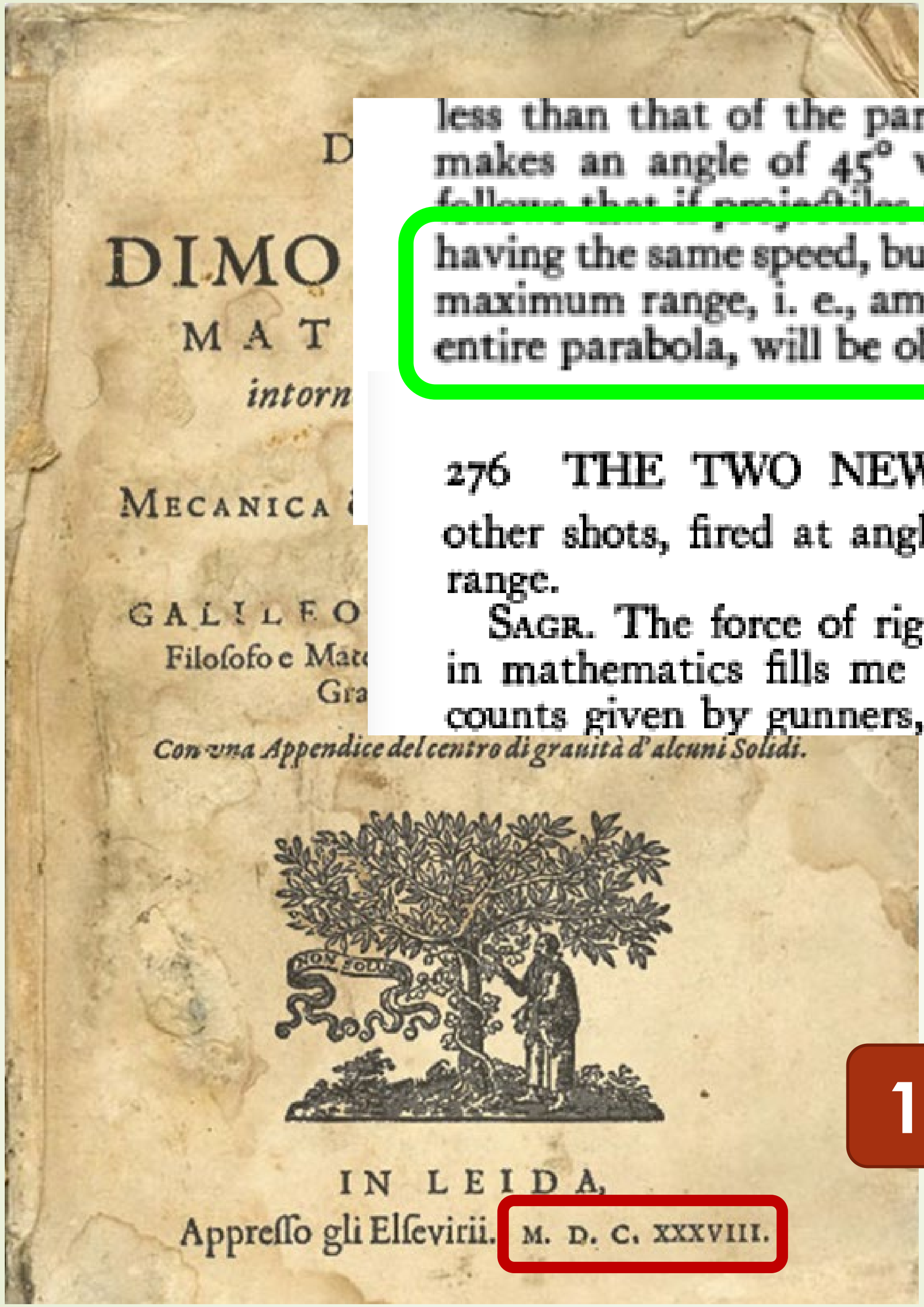

less than that of the parabola *bd*, for which the tangent at *d* makes an angle of 45° with the horizontal. From which it follows that if projectiles are fired from the terminal point *d*, all having the same speed, but each having a different elevation, the maximum range, i. e., amplitude of the semi-parabola or of the entire parabola, will be obtained when the elevation is 45°: the

other shots, fired at angles greater or less will have a shorter range.

SAGR. The force of rigid demons
in mathematics fills me with won
counts given by gunners, I was alr

1638

Dialogues Concerning
TWO NEW SCIENCES
GALILEO GALILEI

TRANSLATED BY
Henry Crew & Alfonso de Salvio

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Last century knowledge:

There seems to be a simple "puzzle"…

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Maximizing the range of the shot put

D. B. Lichtenberg and J. G. Wills
Physics Department, Indiana University, Bloomington, India
(Received 12 July 1976)

We address the problem of the optimum angle at w
order to achieve maximum distance.

I. OPTIMAL ANGLE NEGLECTING AIR RESISTANCE

In most elementary physics textbooks, it is stated that the maximum range of a projectile is attained when the angle of firing is 45°. However, film studies by Cureton[1] of shot putters in action have shown that the best putters release the shot at an angle of between 40° and 42° from the horizontal. We have spoken to several people, both in physics and in coaching, and they seemed genuinely puzzled by the discrepancy. (A common reaction among physicists was to attribute the difference to the effect of air resistance, but as we shall show in Sec. II, air resistance does not appreciably affect the optimum angle.)

$$0 = h + R\tan\theta - (½)gR^2 \text{se}$$

Solving for R, we get

$$R = v^2\cos\theta[\sin\theta + (\sin^2\theta + 2gh$$

By inspection of Eq. (6), we see that inc
increase R. Thus, as expected, the putt
maximize both the speed and height of
To find the maximum range as a fun
take the derivative of R with respect to θ
zero. We obtain

$$\sin^2\theta_m = (2 + 2gh/v^2)^{-}$$

American Journal of Physics **46**, 546 (1978) doi: 10.1119/1.11258

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Horizontal Range of a Projectile

Resolution:

Shot (ball) is released some h $(\sim 7')$ *above* where it lands.

$0 = v_{0y}T$ ...

$v_{0y} - \frac{1}{2} g$ ...

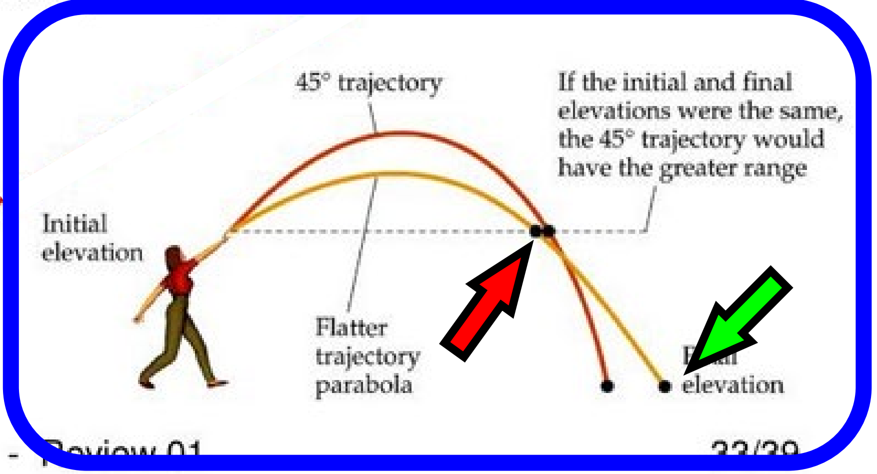
$T = \frac{2v_{0y}}{g}$ ...

$$R = v_{0x}T = (v_0 \cos\theta_0)\left(\frac{2v_0}{g}\sin\theta_0\right) = \frac{v_0^2}{g} 2\sin\theta_0\cos\theta_0$$

$$2\sin\theta_0\cos\theta_0 = \sin 2\theta_0$$

$$\boxed{R = \frac{v_0^2}{g}\sin 2\theta_0}$$

R is maximum when θ_o=45°, so that sin $2\theta_o$ = 1.



https://image3.slideserve.com/6601227/horizontal-range-of-a-projectile-l.jpg

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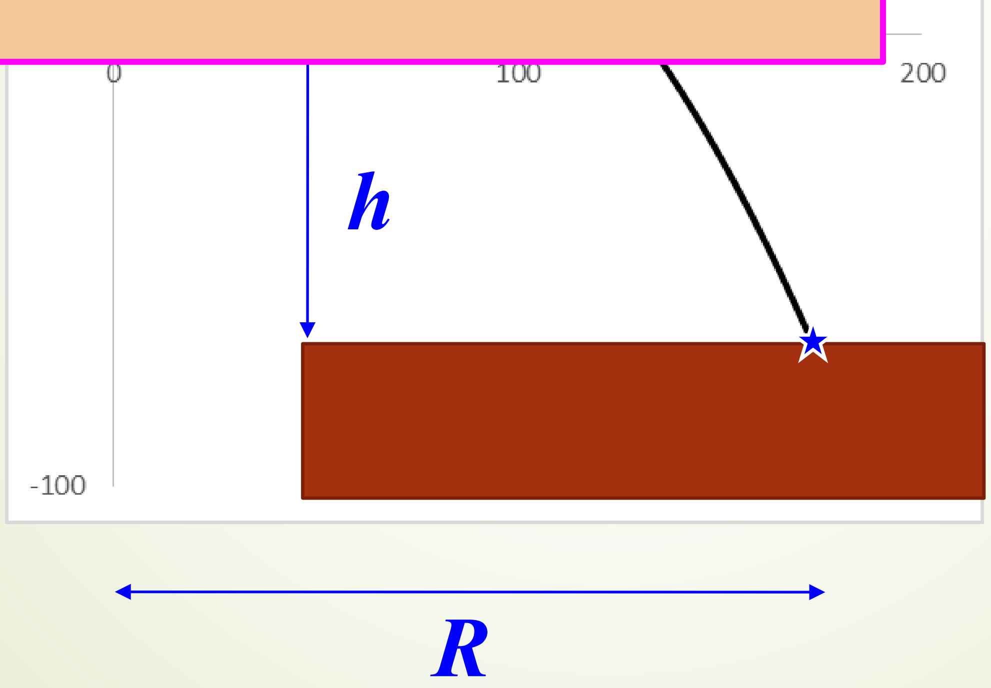
(x,y)

- *Find* $R(\theta)$

  Set $dR/d\theta = 0$

  Solve for θ_{opt}



Given v *and* h,

what θ

maximizes R ?

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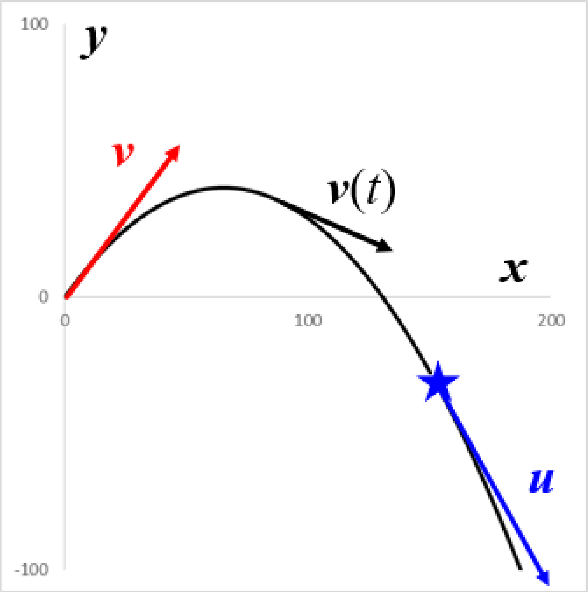
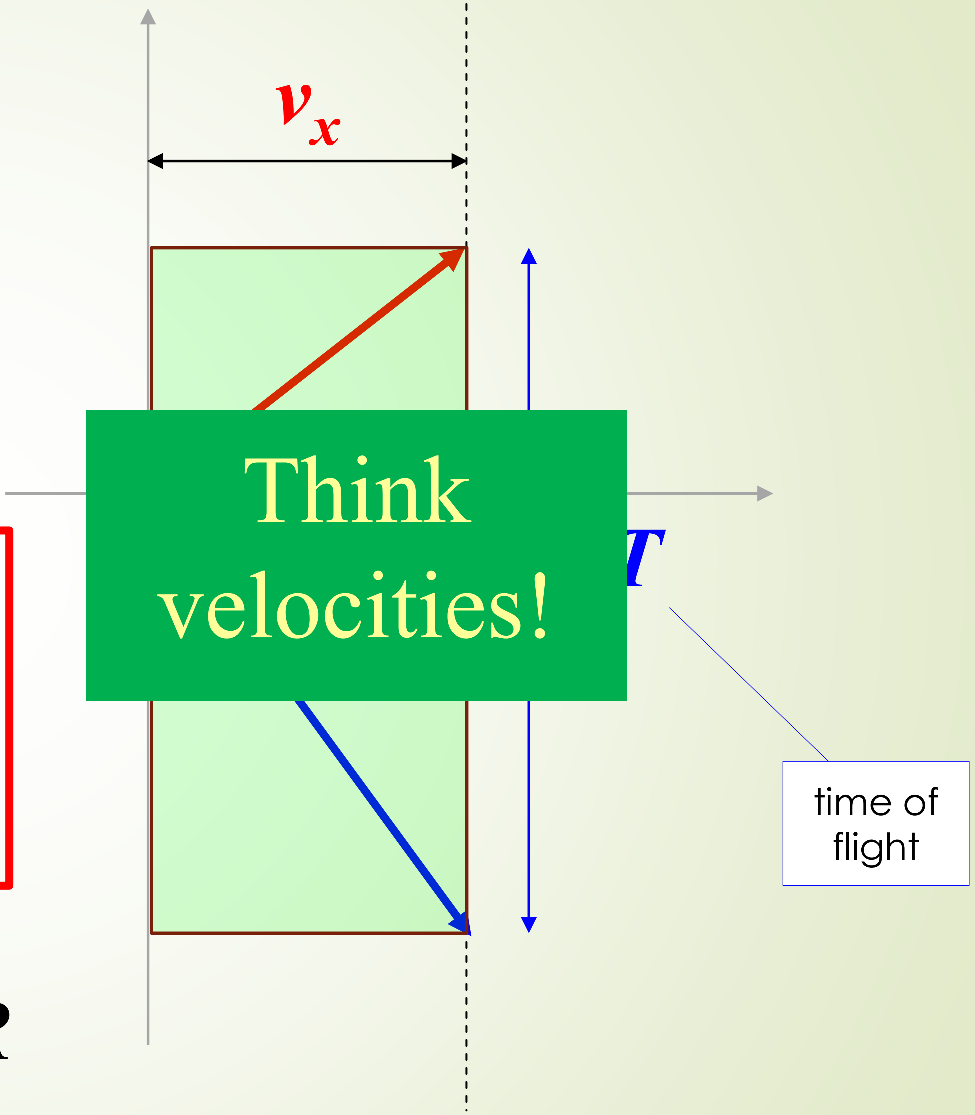


$$\text{"}Area\text{"} = v_x g T$$
$$= gR$$



"*Area*" of rectangle $\propto R$

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In other words, the maximum ***range*** problem

…is just the maximum ***area*** problem,

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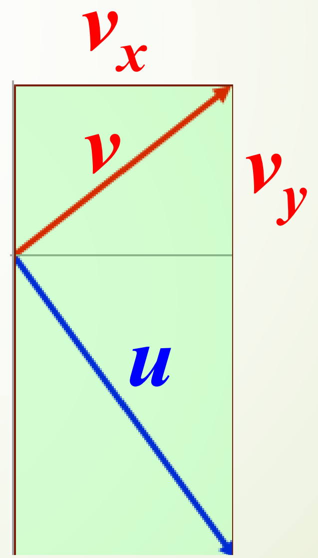
In other words, the maximum ***range*** problem

…is just the maximum ***area*** problem,

subjected to

Fixed launch speed $\Leftrightarrow$ fixed length of red line;

Fixed landing speed $\Leftrightarrow$ fixed length of blue line.



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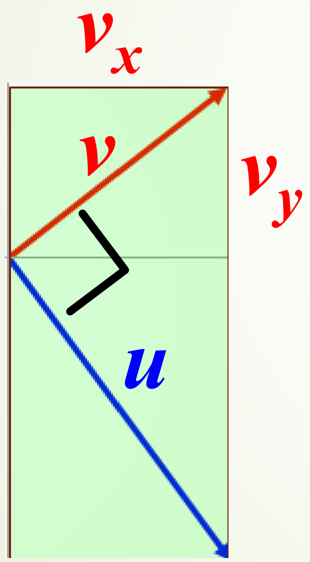
From a geometry detour, we conclude that

in the maximum ***range*** trajectory, the

initial velocity

is orthogonal to the

final velocity !!



…which readily leads to the L&W result:

To find the maximum range as a function of θ, we must take the derivative of R with respect to θ and set it equal to zero. We obtain

$$\sin^2\theta_m = (2 + 2gh/v^2)^{-1}, \qquad (7)$$

where θ_m is the angle which gives the maximum range R_m.

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Other points of interest

- "Sophisticated vector methods"
- "Envelope"
- "Duality"

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- If your students know vector methods…

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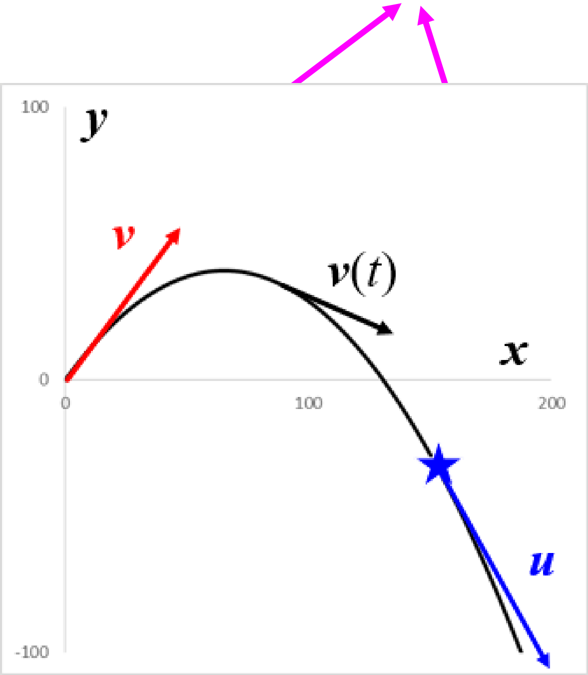
- If your students know vector methods…

Consider $v \times u$



Maximizing the ra … t calculus

P. Palffy-Muhoray
Department of Physic … ıncouver, British Columbia, Canada

D. Balzarini
Department of Physic … itish Columbia, Canada

(Receiv d 3 Novem … ı 1981)

American Journal of Physics **50**, 181 (1982)

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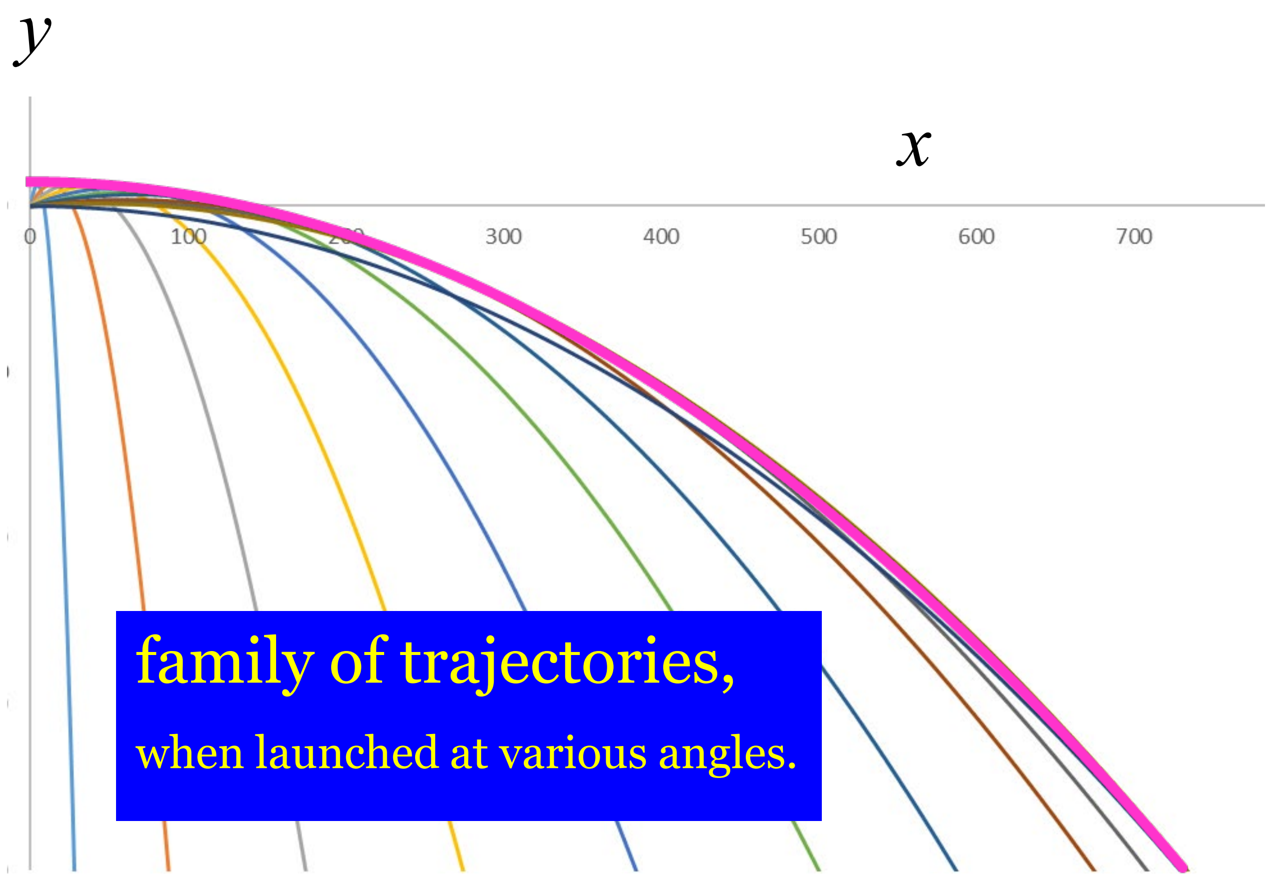
“Envelope”

Edge of region in x-y plane where projectile can reach...



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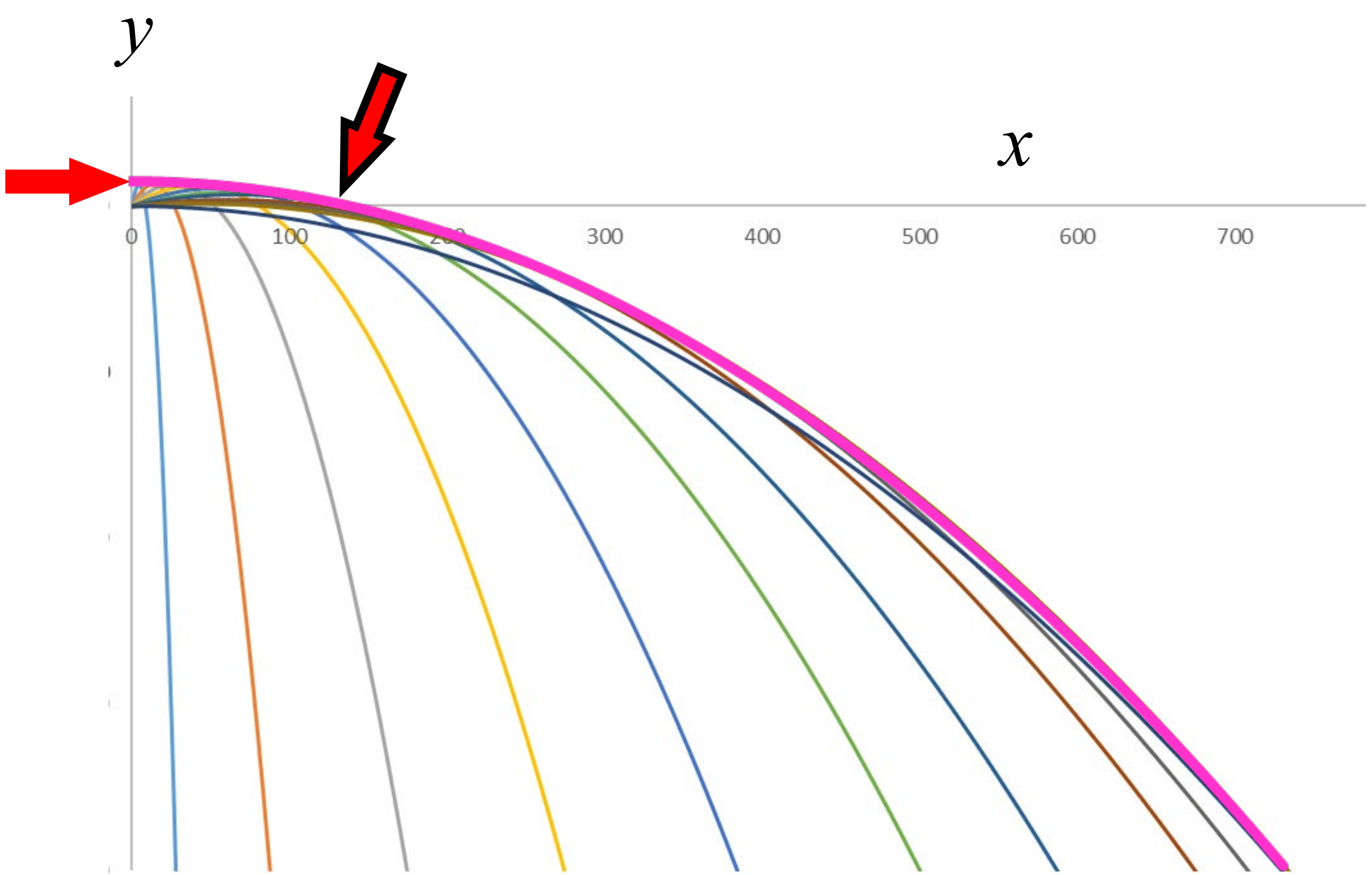
“Envelope”

$gH \equiv v^2/2$

Edge of region in x-y plane where projectile can reach... is also a simple quadratic: $y = H\,[1 - (x/2H)^2\,]$!



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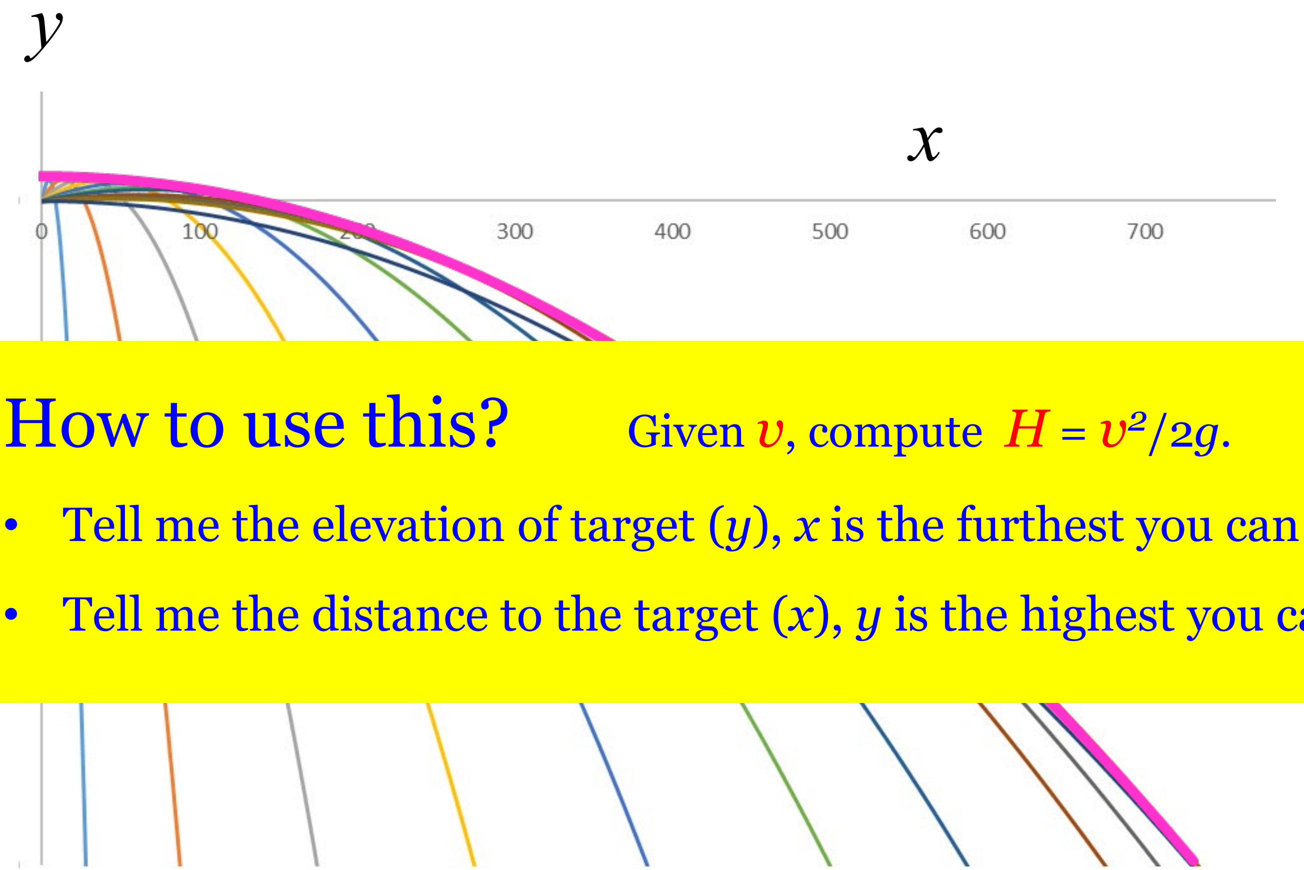
"Envelope"

$$\boldsymbol{y}_{envelope}(\boldsymbol{x}) = H - \boldsymbol{x}^2/4H$$



How to use this?

Given v, compute $H = v^2/2g$.

- Tell me the elevation of target (y), x is the furthest you can reach.
- Tell me the distance to the target (x), y is the highest you can get.

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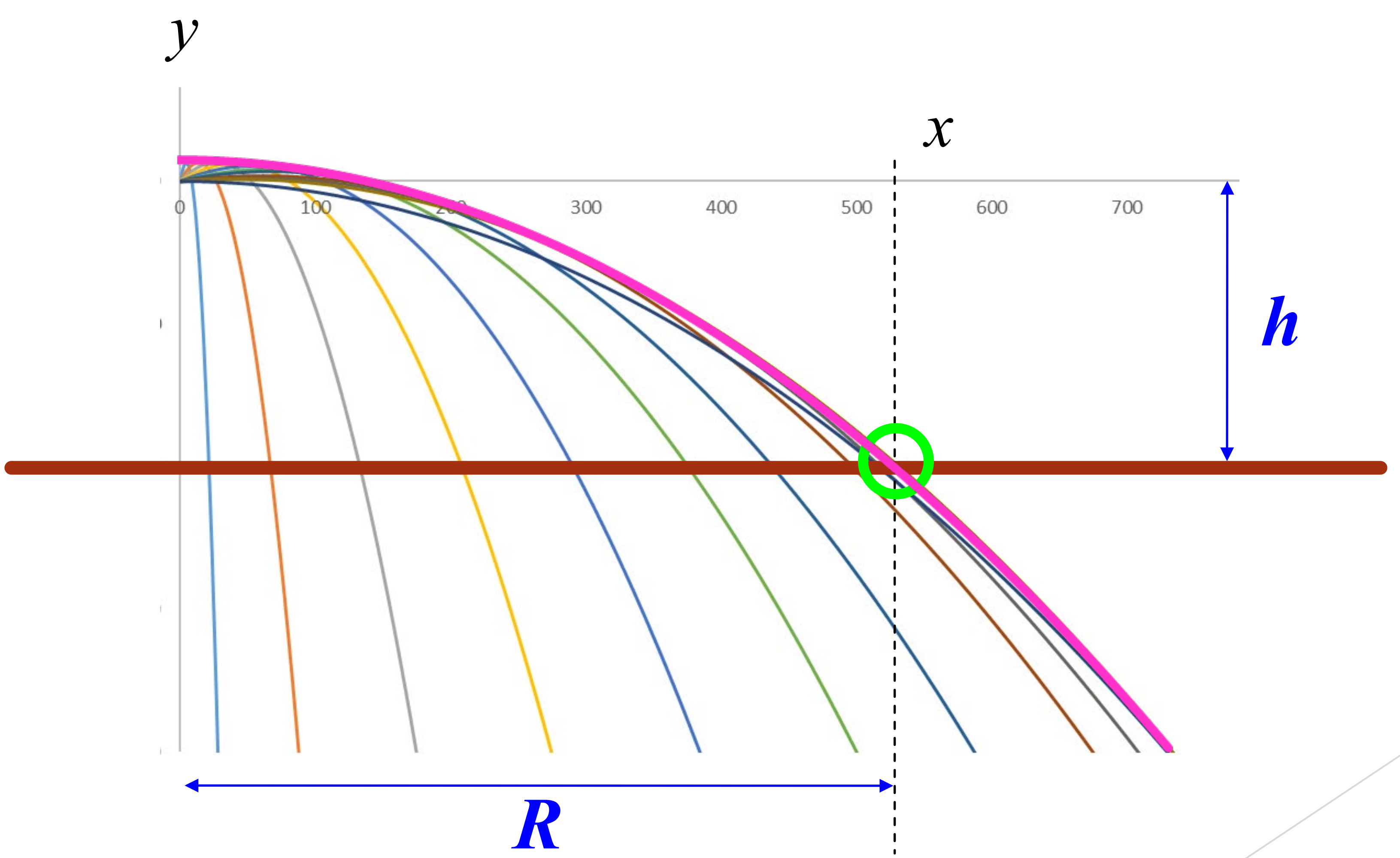
“Envelope”

<u>Original question</u>: Target is a distance ***h*** *<u>below</u>*, what’s the ***furthest*** (***R***) you can reach?



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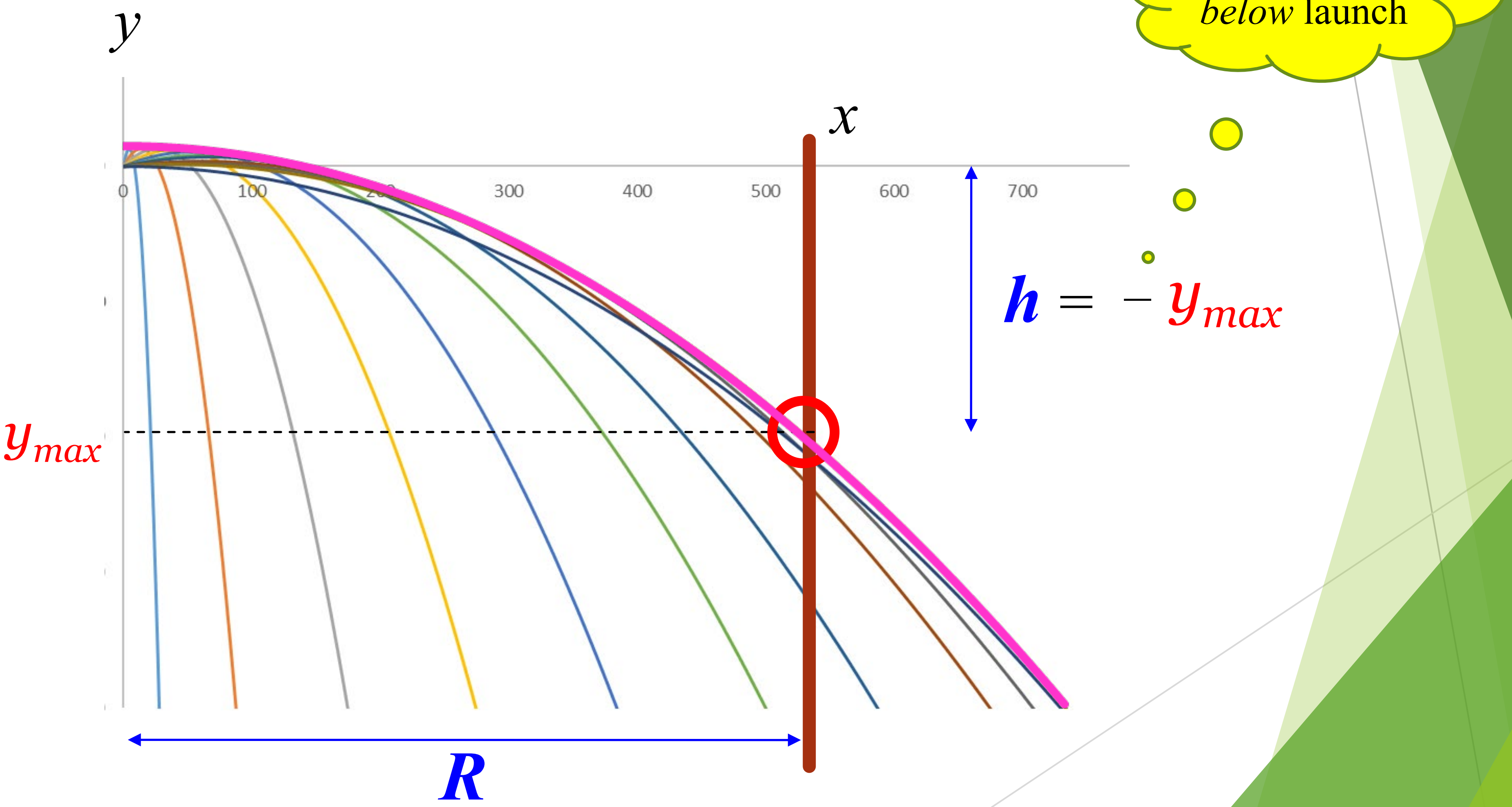
"Envelope"

Rephrase to: ***Given*** a wall a distance ***R*** away, what's the ***highest*** point (y_{max}) you can reach?



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"Envelope"

$$x(t) = v_x t \;\; ; \quad y(t) = v_y t - (g/2)t^2$$

- Eliminate $t = x/v_x$
- Use $v_y/v_x = tan\,\theta$ and define by simpler symbol τ
- Write $v_x{}^2 = v^2 cos^2\theta \;\Rightarrow\; v_x{}^{-2} = v^{-2}(1+\tau^2)$
- Simplify combination

$$2gv^{-2} \equiv 1/H$$

$$\boxed{y(x,\tau) \;=\; \tau x \;-\; (1/4H)(1+\tau^2)x^2}$$

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$$\tau \equiv \tan\theta$$

"Envelope"

$$\boldsymbol{y}(\boldsymbol{x},\tau) = \tau \boldsymbol{x} - (1/4H)(1+\tau^2)\boldsymbol{x}^2$$

Original problem:

- Set $y = -h$ and find $x = R(\tau)$ from above (quadratic! which root?)
- Differentiate w.r.t. θ (ouch!) and set that to zero to find θ_{opt} (***OUCH!!***) .

Rephrased problem:

- Set $x = R$ and find $-h = y(\tau)$ from above (right there, nothing to solve!)
- Differentiate w.r.t. τ (easy!) and set that to zero (trivial!) to find

$$1 = (R/2H)\,\tau_{opt} \;\Rightarrow\; R\tau_{opt} = 2H\,.$$

- Substitute $\tau \boldsymbol{x} \Rightarrow R\tau_{opt}$ back (easy!) to get $y_{env}(x)$

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Take-home messages (or "Morals")

a.k.a.

TIPS to make life easier!

TIPS you can teach your students!

TIPS to share with your friends/colleagues!

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Take-home messages

- Don't just follow the requests (questions) literally.

  Here, you would find $R(\square)$, write out $dR/d\square = 0$ and try to solve that for $\square_{opt}$.

  All these steps are "hard" – or at least prone to errors.

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Take-home messages

- Don't just follow the requests (questions) literally.

  Here, you would find $R(\square)$, write out $dR/d\square = 0$ and try to solve that for $\square_{\text{opt}}$.

  All these steps are "hard" – or at least prone to errors.

- Look for equivalent formulations (rephrasing) of the question. They may turn out to get you down an easier path.

  Here, I showed you that maximizing R given h is the same as fixing R and maximizing h.

  Finding $R(\square;h)$ is messier than finding $h(\square;R)$ in this case.

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Take-home messages

- Don't just follow the requests (questions) literally.

  Here, you would find R(□), write out dR/d□= 0 and try to solve that for □$_{opt}$.

  All these steps are "hard" – or at least prone to errors.

- Look for equivalent formulations (rephrasing) of the question. They may turn out to get you down an easier path.

  Here, I showed you that maximizing R given h is the same as fixing R and maximizing h.

  Finding R(□;h) is messier than finding h(□;R) in this case.

- Look for better variables than the ones given. They may turn out …

  If we face an optimization condition like dy/dx=0…

  By changi … " than $y(x)$.

  Now, we l … But, if

  you *choose a variable* with dz/dx□0 (in the region of interest),

  then your opt.cond. is just dy/dz=0 (which may be "nicer").

Here, τ is much better than □.

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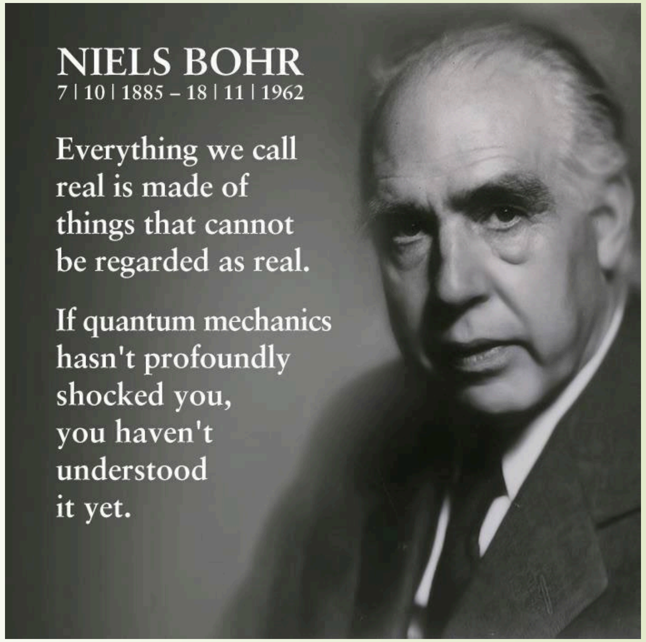
"Duality"

When he got "knighted," he got to design his ***<u>Coat of Arms.</u>***



wave-*particle* duality

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"Duality"

- … ***not*** a precise concept
- Examples abound in mathematics and physics.
- Typically links ***two*** aspects of one problem (or an object) in some complementary manner…
- …its specific role comes from the precise definition of "complementarity."

$x \leftrightarrow -x$; $x \leftrightarrow 1/x$;

x *space* $\leftrightarrow$ k *space*

wave $\leftrightarrow$ *particle* ; $T > T_c \leftrightarrow T < T_c$; $AdS \leftrightarrow CFT$ …

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“Duality” here:

- Two problems: basketball & shot-put

  and

- Two points on a parabola with *orthogonal* slopes

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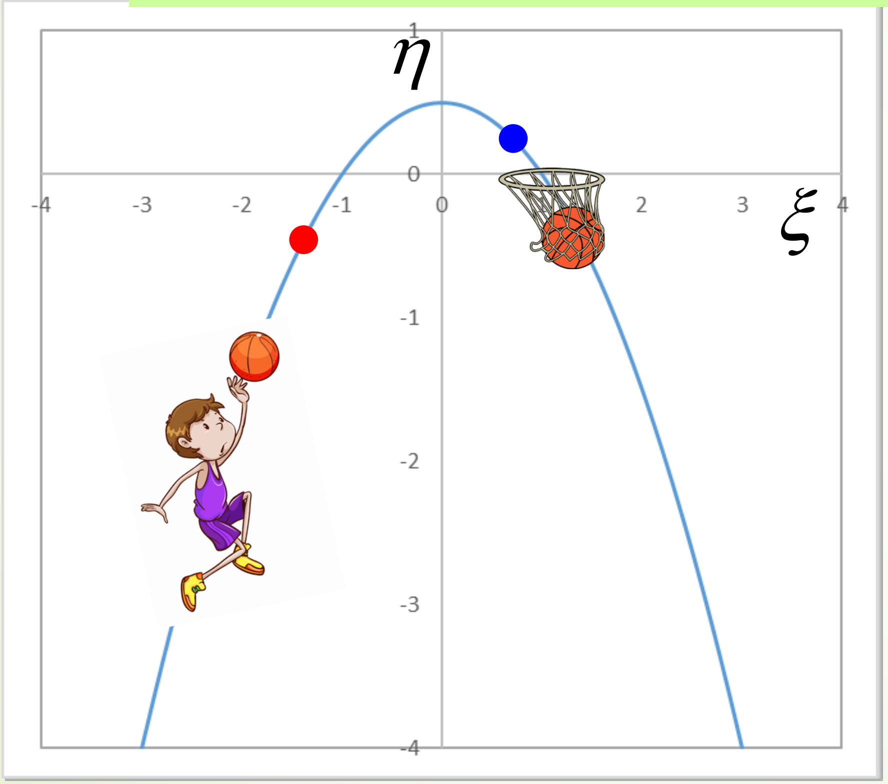
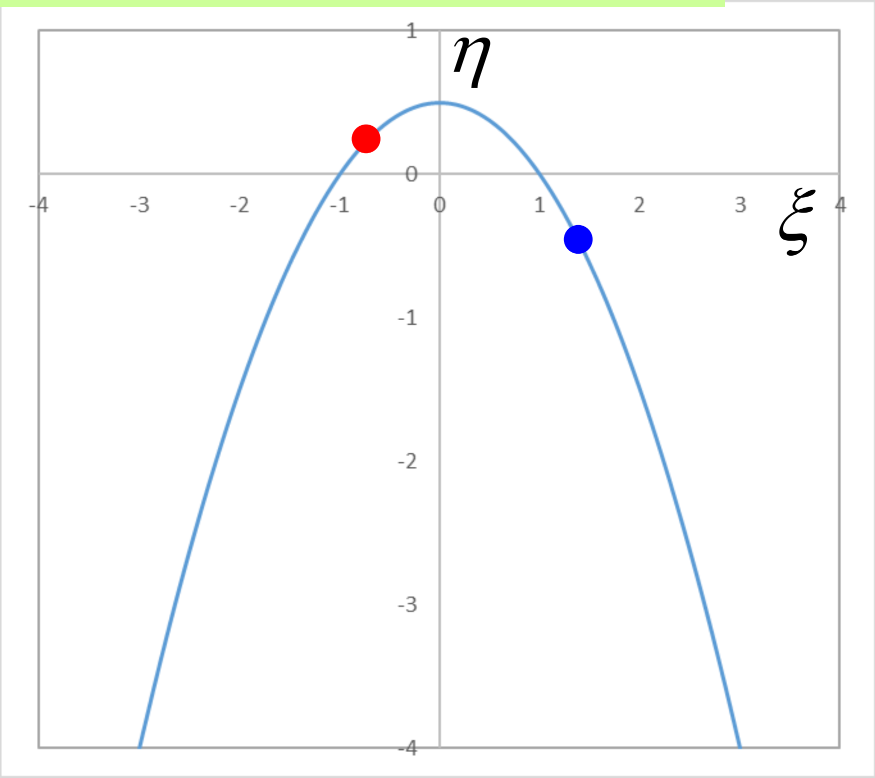
"Duality":

Two problems (basketball & shot put) are dual to each other!

To use technical jargon, via "reflection & time reversal".



basketball



shot put

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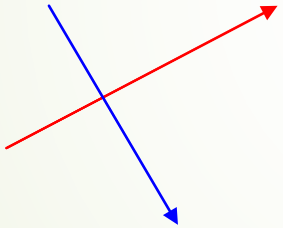
“Duality” here:

Two points on a parabola with *orthogonal* slopes.

Given $y(x)$, slope at x is dy/dx □ s

… slope at x is dy/dx □ s



The two are *orthogonal* if

$s = -1/s$ i.e.,

$ss = -1$

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“Duality” here:

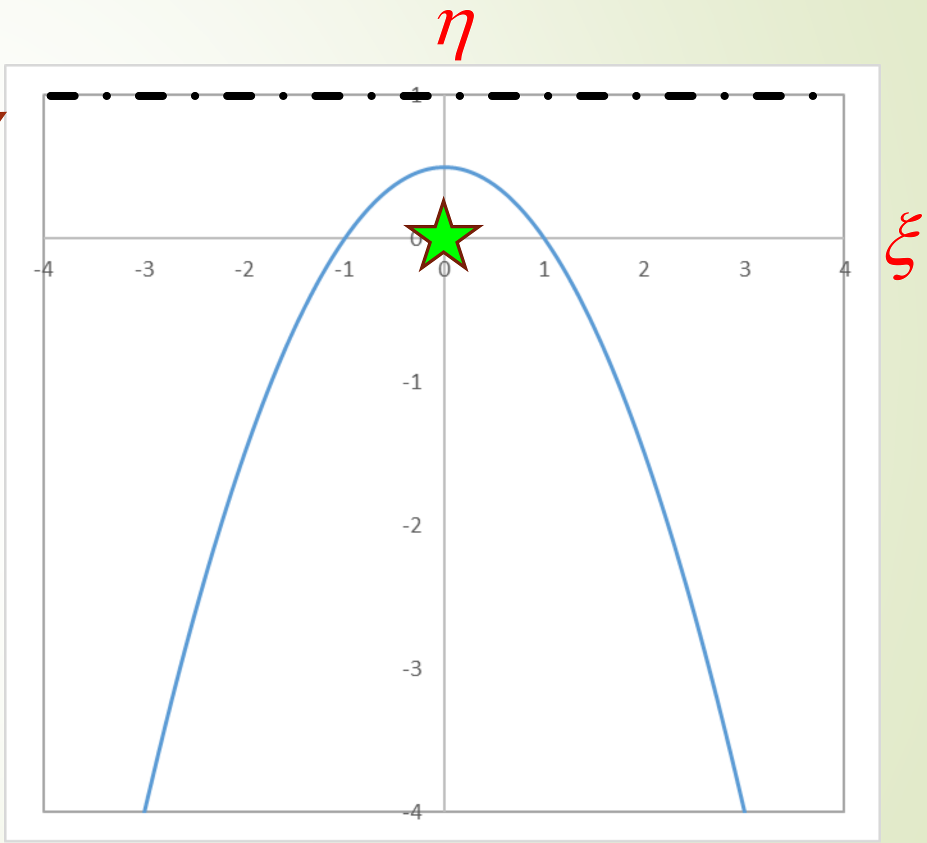
Two points on a parabola with *orthogonal* slopes.

Standard parabola:

- focus ☆ at 0
- directrix at 1
- curvature ±1
- intercepts at ±1

$$\eta = \frac{1 - \xi^2}{2}$$

- slope at ξ : $-\xi$



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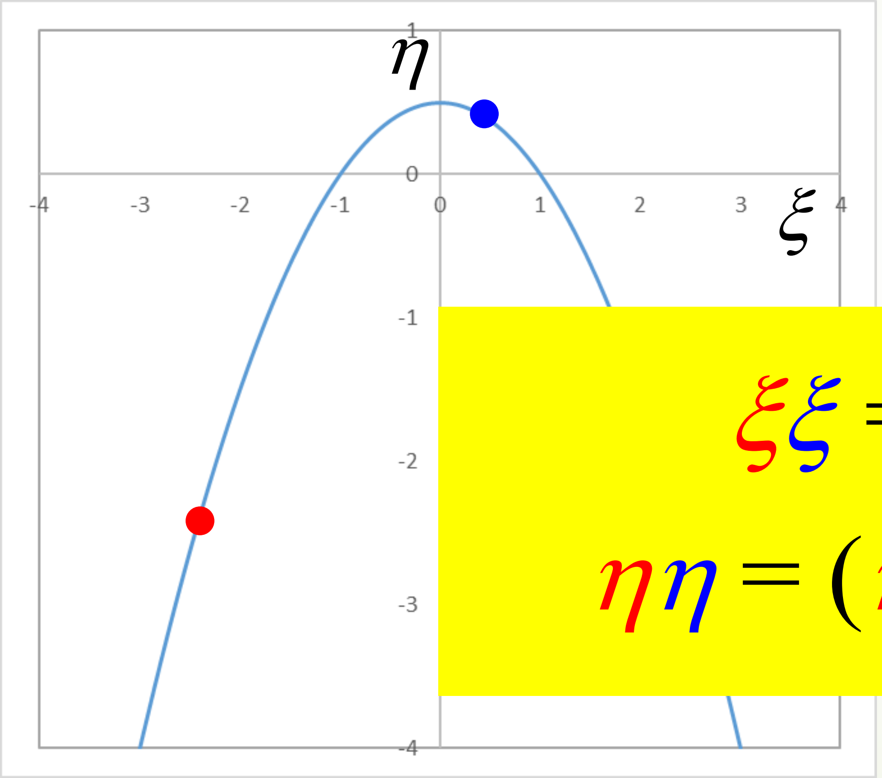
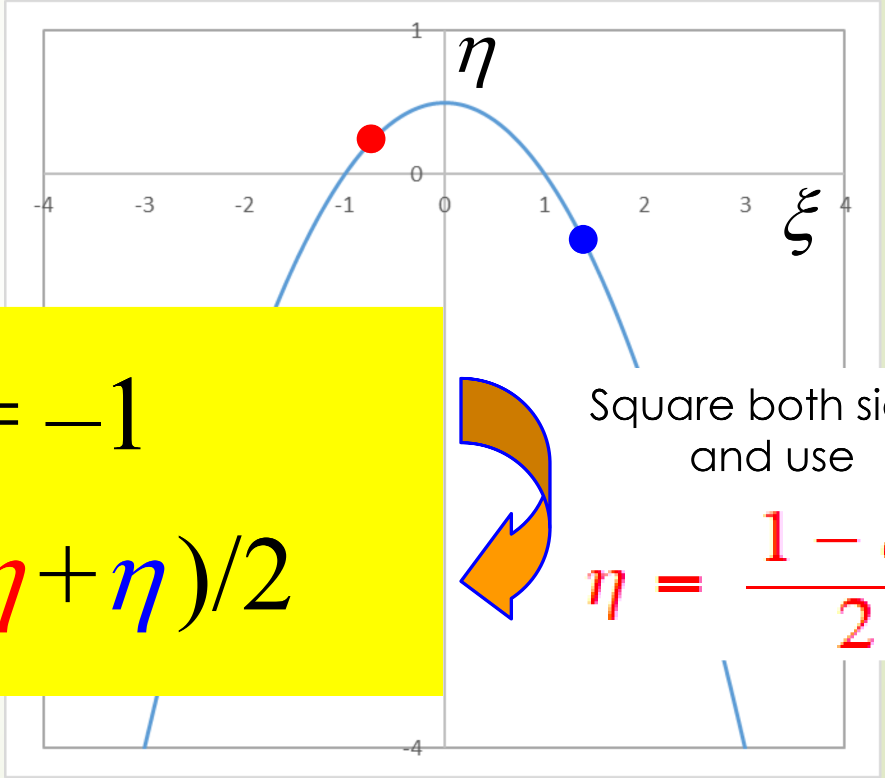
“Duality”:

Two points on a standard parabola with *orthogonal* slopes, e.g.





$$\xi\xi = -1$$

$$\eta\eta = (\eta + \eta)/2$$

Square both sides and use

$$\eta = \frac{1-\xi^2}{2}$$

basketball

shot put

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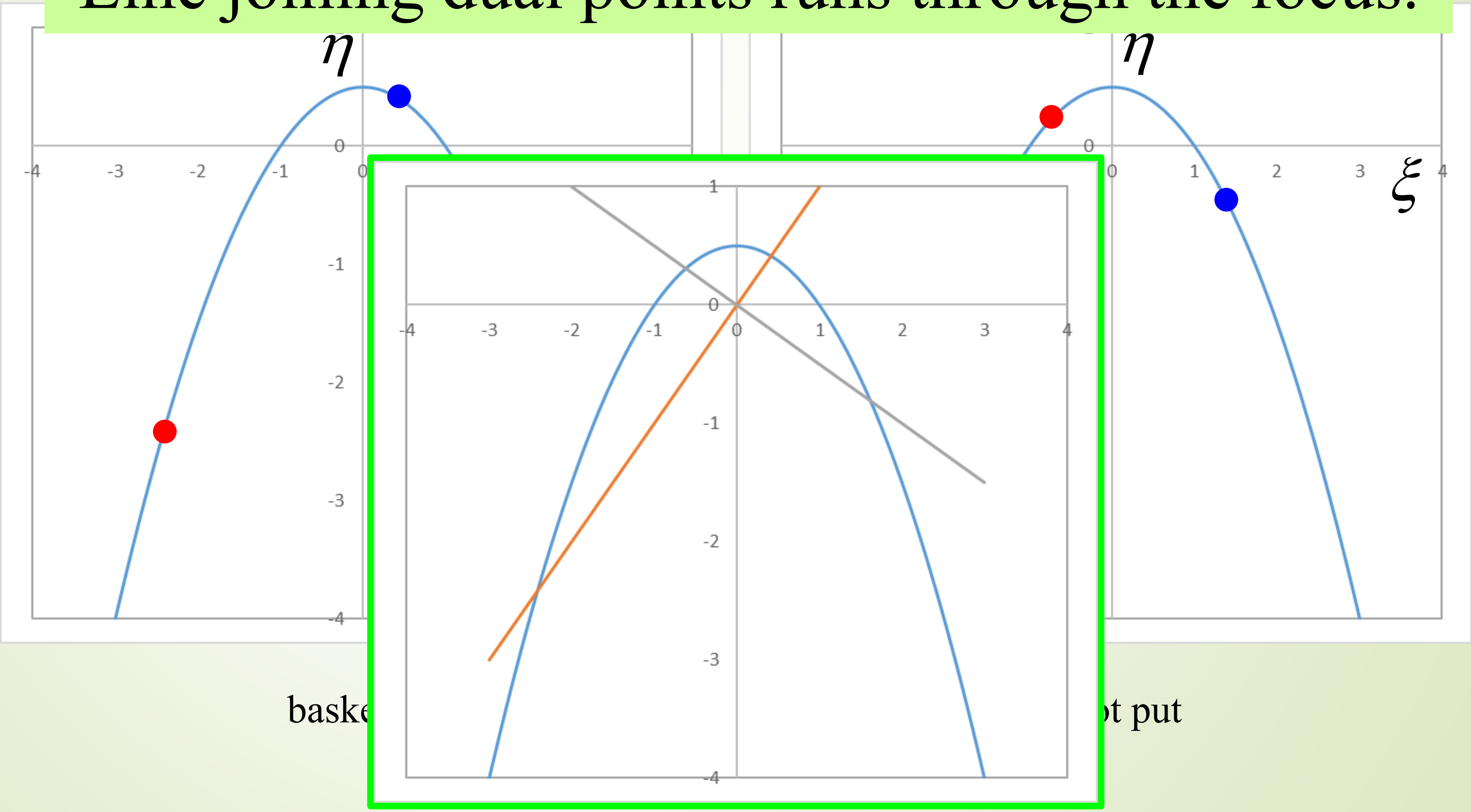
“Duality”:

Two points on a standard parabola with *orthogonal* slopes,

e.g.

Line joining dual points runs through the focus.



baske … t put

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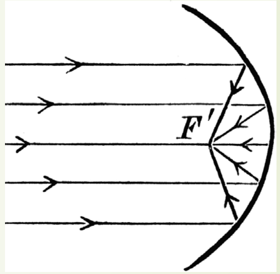
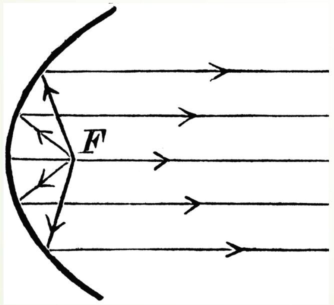
“Duality”:

Switching gears drastically… to geometric optics.

Consider a parabolic mirror (2-D would do)

Beams parallel to the axis impinging on the mirror are reflected to the focus.





Light beams emerging from the focus are reflected to rays parallel to the axis.

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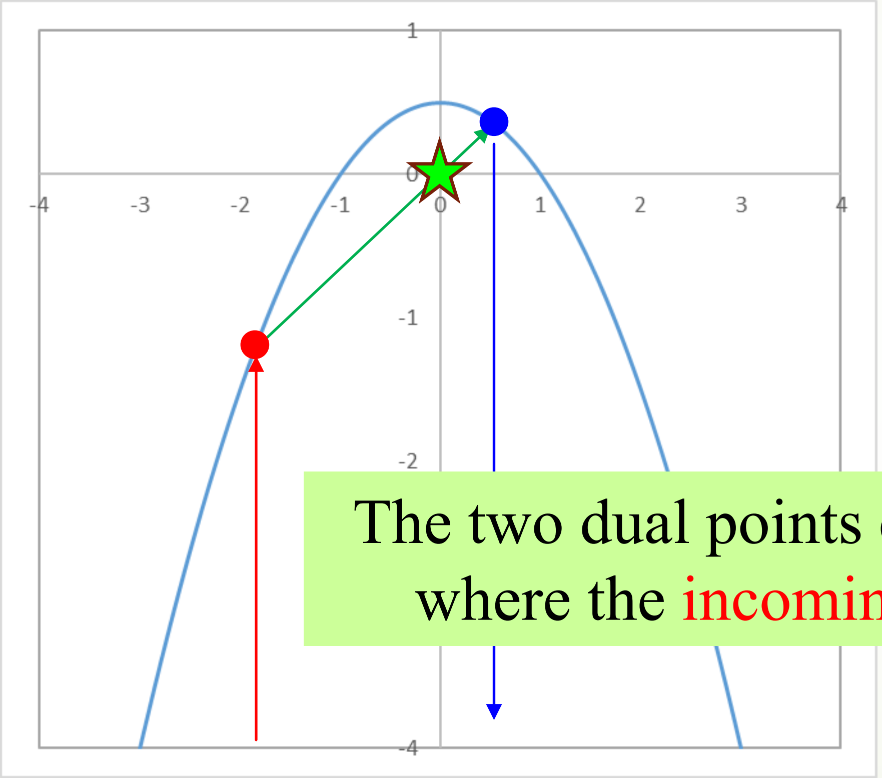
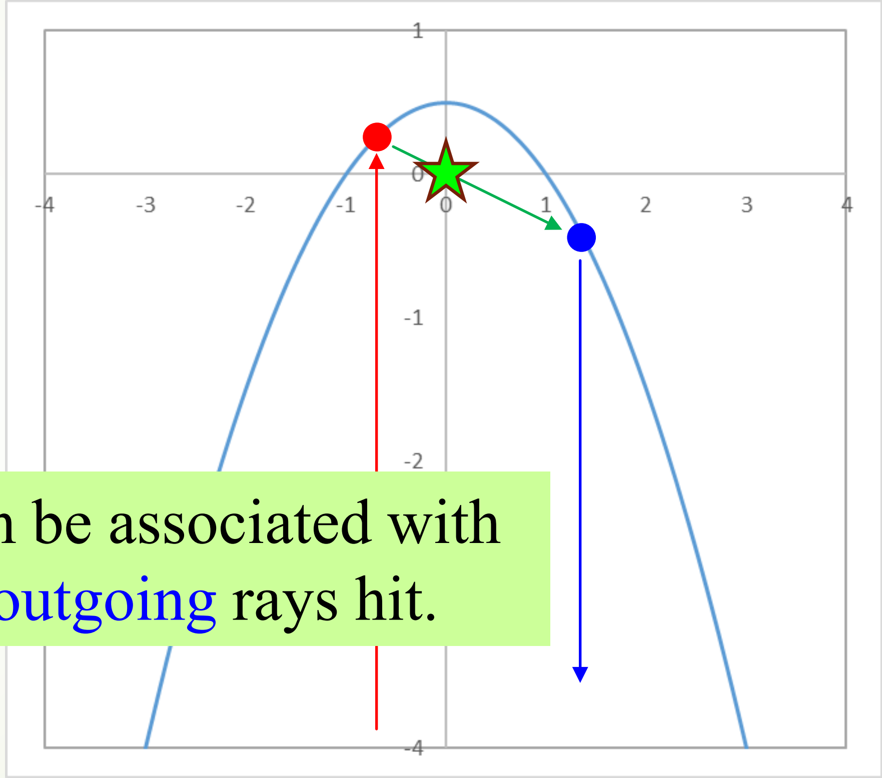
“Duality”:

Switching gears drastically… to geometric optics.





The two dual points can be associated with where the incoming/outgoing rays hit.

basketball

shot put

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Take-home messages

- Look for similarities in problems from different areas/fields.
  They may turn out to be of great help with …

  … providing insights into your problem.
  … offering methods for solving your problem.
  … broadening your perspectives.
  … uncovering deep connections.

- Some examples:

  Here, it's projectile optimization ~ ray optics.
  Elsewhere, there is…

  - distribution of primes ~ $\zeta(z) \equiv \Sigma_n 1/n^z$,
  - superconductivity ~ Higgs mechanism,
  - QFT renormalization ~ critical phenomena universality
    ~ central limit theorem ~ normal distribution,
  - predator prey dynamics ~ onset of (some kind of) turbulence,
  - Quantum Hall effect ~ El Niño

  …

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What else?

There is **a graphic !!** way to determine the optimal trajectory for projectile motion.

It focuses on the role of the *directrix* & *focus*.

The method easily generalizes to the same problem for a *spherical earth*!

… while the analytic method is **HORRENDOUS**!!

Write me a email, if you can't wait till the next CSAAPT meeting.

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Conclusion

Even <u>ancient/elementary</u> problems in physics may lead to a trove of <u>novel/interesting</u> treasures!

Enjoy digging elsewhere !!

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Thank you for coming!

Any questions ?

or comments ?

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Optimal launch angles:

Novel perspectives of an ancient problem

We revisit the ancient problem of finding the optimal angle for launching a projectile so as to maximize the range. In the last meeting, we showed a solution to this problem using only geometry. Here, we present some novel perspectives. One is the notion of "duality" (between the launch and target sites). Another is the envelope of all trajectories (for launching at different angles, but with the same speed). Consequences associated with these ideas are explored. Finally, we present a simple, unified approach that includes launches on a spherical earth. Relying only on energy conservation and properties of ellipses/parabolas, this approach should lie within the grasp of high school students.

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