Computer Project on Orbital Motion

Phuc Tran

Brightpoint Community College

Objective

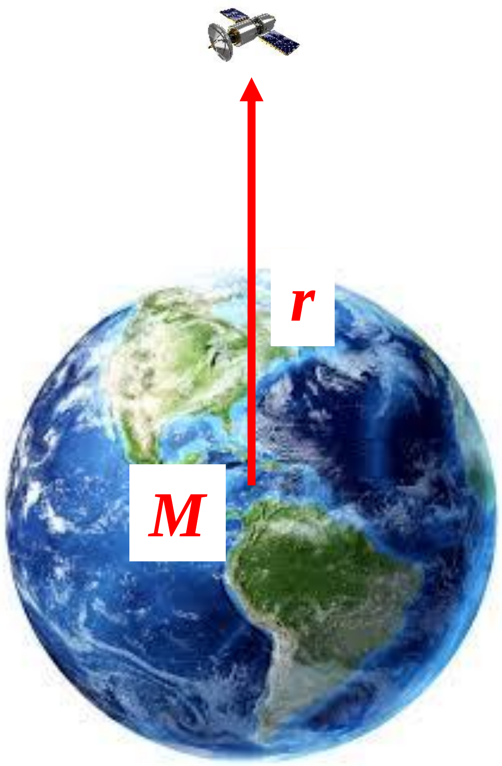
This is part of an effort to introduce students to computational physics.

Overview

Solve the equation of motion numerically.

Compare numerical solution against theoretical orbits.

Equation of Motion



Newton's 2nd law $\mathbf{F} = m\boldsymbol{a}$

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$$

The force is given by the law of universal gravitation

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}$$

$$\longrightarrow \quad m\frac{d\boldsymbol{v}}{dt} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}$$

Numerical Solution

Two coupled equations

$$m\frac{d\boldsymbol{v}}{dt} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}$$

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}$$

Given the initial position and velocity, solve the equations numerically by iteration

Conserved Quantities

Energy

The force of gravity is a conservative force, therefore, the mechanical energy is constant.

$$E = K + U$$

Kinetic energy $$K = \frac{1}{2}mv^2$$

Gravitational potential energy $$U = -\frac{GMm}{r}$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Angular Momentum

Angular momentum is $\mathbf{L} = \boldsymbol{r} \times m\boldsymbol{v}$

Newton 2nd law of rotational motion $\boldsymbol{\tau} = \dfrac{d\mathbf{L}}{dt}$

Torque is $\boldsymbol{\tau} = \boldsymbol{r} \times \mathbf{F} = \boldsymbol{r} \times \left(-\dfrac{GMm}{r^2}\hat{\boldsymbol{r}}\right) = -\dfrac{GMm}{r^2}(\boldsymbol{r} \times \hat{\boldsymbol{r}}) = 0$

$\longrightarrow \quad 0 = \dfrac{d\mathbf{L}}{dt} \quad \longrightarrow \quad \mathbf{L} = constant$

Planar Motion

Without loss of generality, we can assume the initial position and velocity are in the x-y plane, i.e.,

$$\boldsymbol{r} = (x, y, 0)$$

$$\boldsymbol{v} = (v_x, v_y, 0)$$

Equation of Motion

$$m\frac{d\boldsymbol{v}}{dt} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}$$

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}$$

Conserved quantities

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\mathbf{L} = m\boldsymbol{r} \times \boldsymbol{v}$$

Dimensionless Equations

Two Natural Parameters of Problem

Radius of central object R_C

Acceleration due to gravity g $\quad$ $$g = \frac{GM}{R_C^2}$$

Dimensionless time $\quad$ $$t' = \sqrt{\frac{g}{R_C}}\,t$$

Dimensionless length $\quad$ $$r' = \frac{r}{R_C}$$

Dimensionless velocity $\quad$ $$v' = \frac{v}{\sqrt{gR_C}}$$

Dimensionless Equation of Motion

$$\frac{d\boldsymbol{v}}{dt} = -\frac{1}{r^2}\hat{\boldsymbol{r}}$$

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}$$

Constants of motion

$$E = \frac{1}{2}v^2 - \frac{1}{r}$$

$$\mathbf{L} = \boldsymbol{r} \times \boldsymbol{v}$$

Note all variables are dimensionless, i.e., the prime has been dropped.

Iterative Equations

$$\frac{d\boldsymbol{v}}{dt} = -\frac{1}{r^2}\hat{\boldsymbol{r}} \longrightarrow \boldsymbol{v}_f = \boldsymbol{v}_i + \left(-\frac{1}{r^2}\hat{\boldsymbol{r}}\right)\Delta t$$

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} \longrightarrow \boldsymbol{r}_f = \boldsymbol{r}_i + \boldsymbol{v}\Delta t$$

For small time step, the position can be taken to be the initial position, and the velocity can be taken to be the average between the initial and final velocity

$$\boldsymbol{v}_f = \boldsymbol{v}_i + \left(-\frac{1}{{r_i}^2}\widehat{\boldsymbol{r}_i}\right)\Delta t$$

$$\boldsymbol{r}_f = \boldsymbol{r}_i + \frac{\boldsymbol{v}_f + \boldsymbol{v}_i}{2}\Delta t$$

Four Iterative Equations

$$v_{x,n+1} = v_{x,n} - \frac{x_n}{(x_n^2 + y_n^2)^{3/2}}\Delta t$$

$$v_{y,n+1} = v_{y,n} - \frac{y_n}{(x_n^2 + y_n^2)^{3/2}}\Delta t$$

$$x_{n+1} = x_n + (v_{x,n} + v_{x,n+1})\frac{\Delta t}{2}$$

$$y_{n+1} = y_n + \left(v_{y,n} + v_{y,n+1}\right)\frac{\Delta t}{2}$$

Types of Orbit

Quadratic Equation for Turning Points

$$E = \frac{1}{2}v^2 - \frac{1}{r}$$

$$\mathbf{L} = \boldsymbol{r} \times \boldsymbol{v} = rv\sin\theta$$

At closest or farthest point in the orbit, $\boldsymbol{r}$ and $\boldsymbol{v}$ are perpendicular so $L = rv \sin 90 = rv$, or

$$v = \frac{L}{r}$$

$$2Er^2 + 2r - L^2 = 0$$

Energy and angular momentum are constant and can be calculated from initial conditions.

Shape of Orbit

Solution to the quadratic equation only gives the turning points of the orbit and not the shape of the orbit.

That would require math beyond first year calculus.

Solution to Quadratic Equation

$$r_{\pm} = \frac{-1 \pm \sqrt{1 + 2EL^2}}{2E}$$

This equation tells us everything about the orbit, from circular to elliptical to parabolic to hyperbolic.

For circular orbit (bound), the two solutions are positive and equal.

For elliptical orbit (bound), the two solutions are positive and unequal.

For parabolic orbit (unbound), one solution is positive, and one solution is infinity.

For hyperbolic orbit (unbound), one solution is positive, and one solution is negative.

Conic Section

Equation of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a^2 - b^2 = c^2$$

Equation of a hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$a^2 + b^2 = c^2$$

Equation of a parabola

$$y = \frac{x^2}{4a}$$

Circular Orbit

For circular orbit (bound), the two solutions are positive and equal.

$$r_\pm = \frac{-1 \pm \sqrt{1 + 2EL^2}}{2E}$$

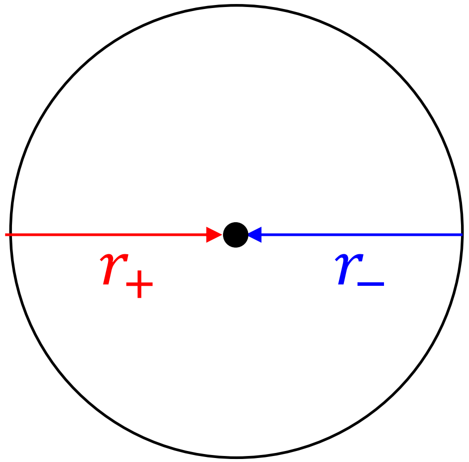
$$1 + 2EL^2 = 0$$

$$r_\pm = -\frac{1}{2E} > 0$$

$$E = -\frac{1}{2r_\pm}$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

$$a = r_\pm = -\frac{1}{2E}$$



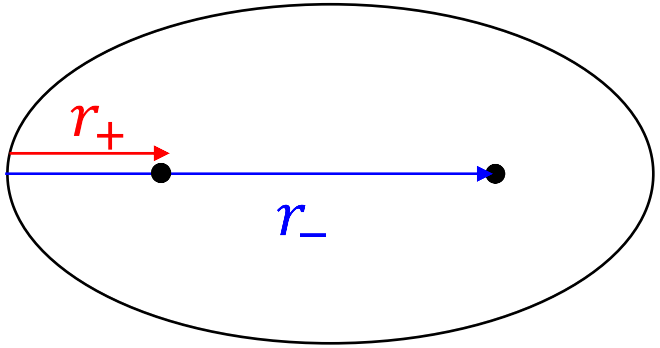
Elliptical Orbit

For elliptical orbit (bound), the two solutions are positive and unequal.

$$r_{\pm} = \frac{-1 \pm \sqrt{1 + 2EL^2}}{2E}$$

This can only be true if the energy is negative, $E < 0$.

$$r_- > r_+ > 0$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a^2 - b^2 = c^2$$

$$a = \frac{|r_+ + r_-|}{2} = \frac{1}{|2E|}$$

$$c = \frac{|r_+ - r_-|}{2}$$

Parabolic Orbit

For parabolic orbit (unbound), one solution is positive, and one solution is infinity.

$$r_{\pm} = \frac{-1 \pm \sqrt{1 + 2EL^2}}{2E}$$

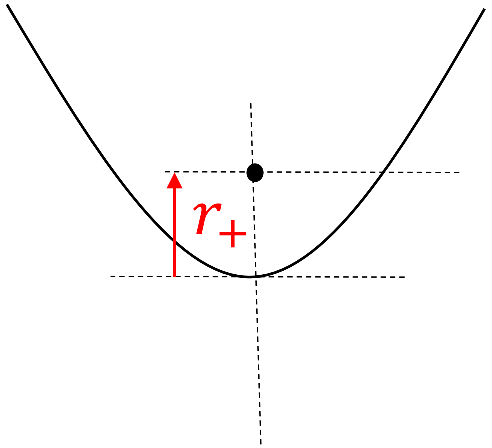
$$y = \frac{x^2}{4a}$$

$$a = r_+$$

This can only be true if the energy is $E = 0$.

$$r_- = \infty$$

$$r_+ = \lim_{E \to 0} \frac{-1 \pm \sqrt{1 + 2EL^2}}{2E} = \frac{L^2}{2}$$



Hyperbolic Orbit

For hyperbolic orbit (unbound), one solution is positive, and one solution is negative.

$$r_\pm = \frac{-1 \pm \sqrt{1 + 2EL^2}}{2E}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

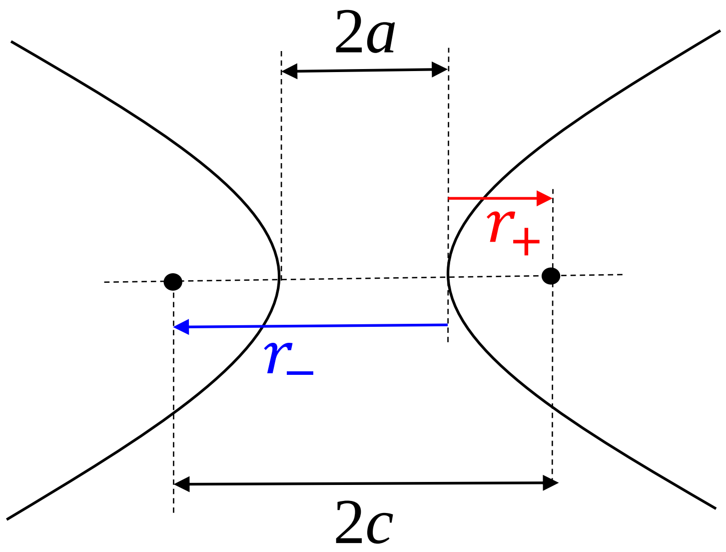
This can only be true if $E > 0$.

$$a^2 + b^2 = c^2$$

$$r_- < 0$$

$$r_+ > 0$$

$$|r_-| > |r_+|$$



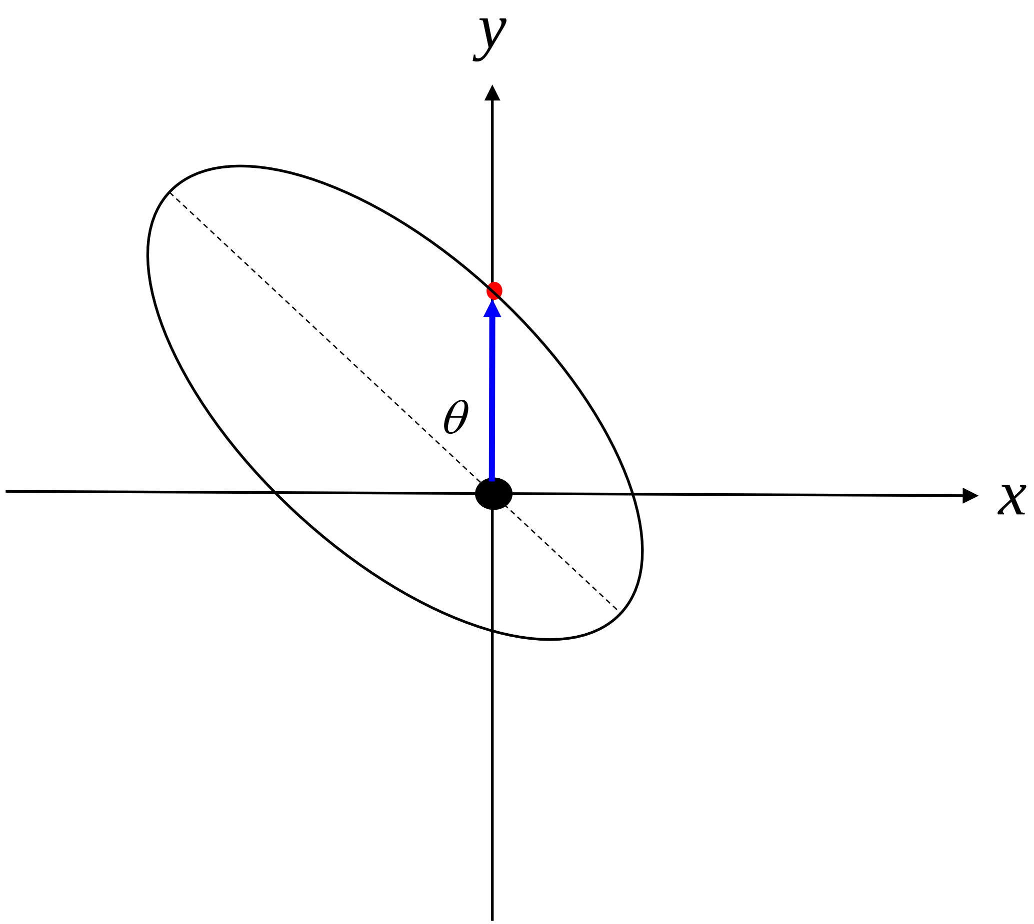
$$a = \frac{|r_+ + r_-|}{2} = \frac{1}{|2E|}$$

$$c = \frac{|r_+ - r_-|}{2}$$

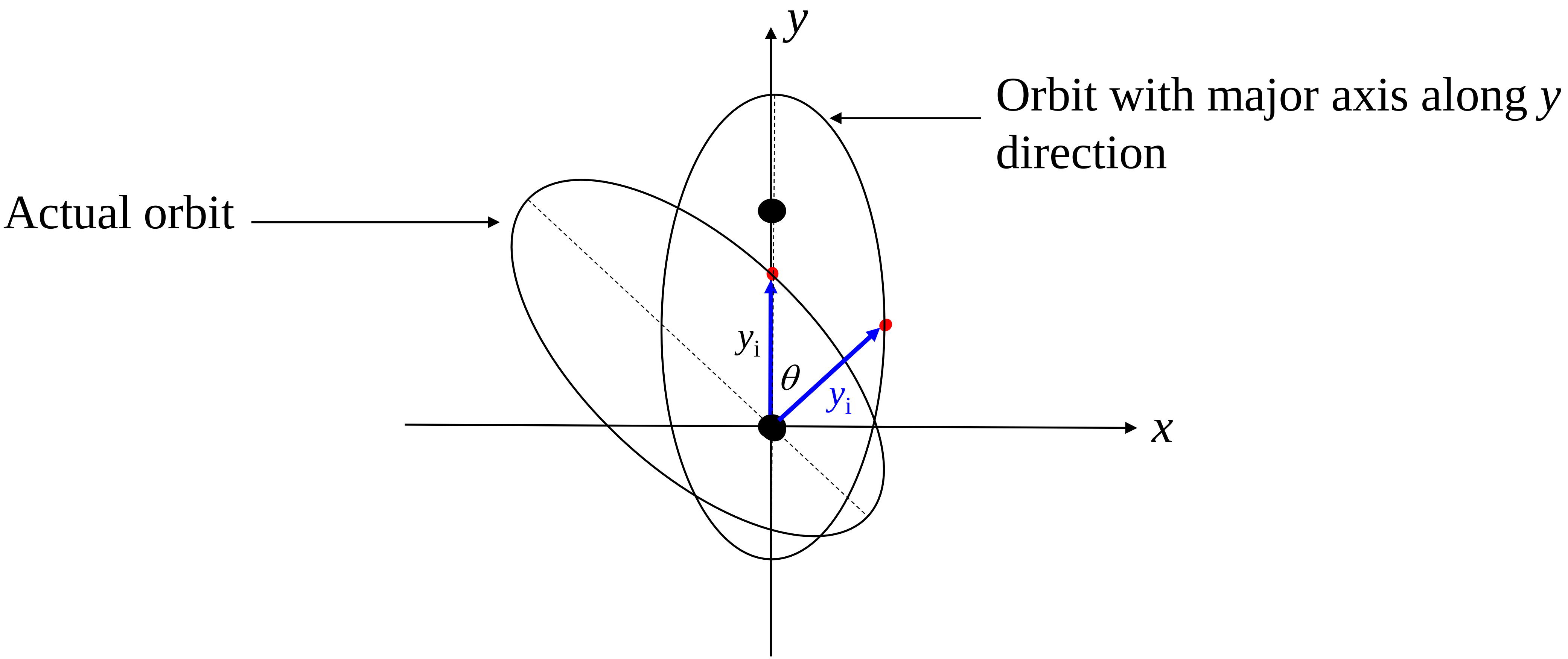
Orientation of Orbit

It is assumed that the initial position is on the *y* axis.

The initial position determines the orientation of the orbit.



Elliptical Orbit Rotation

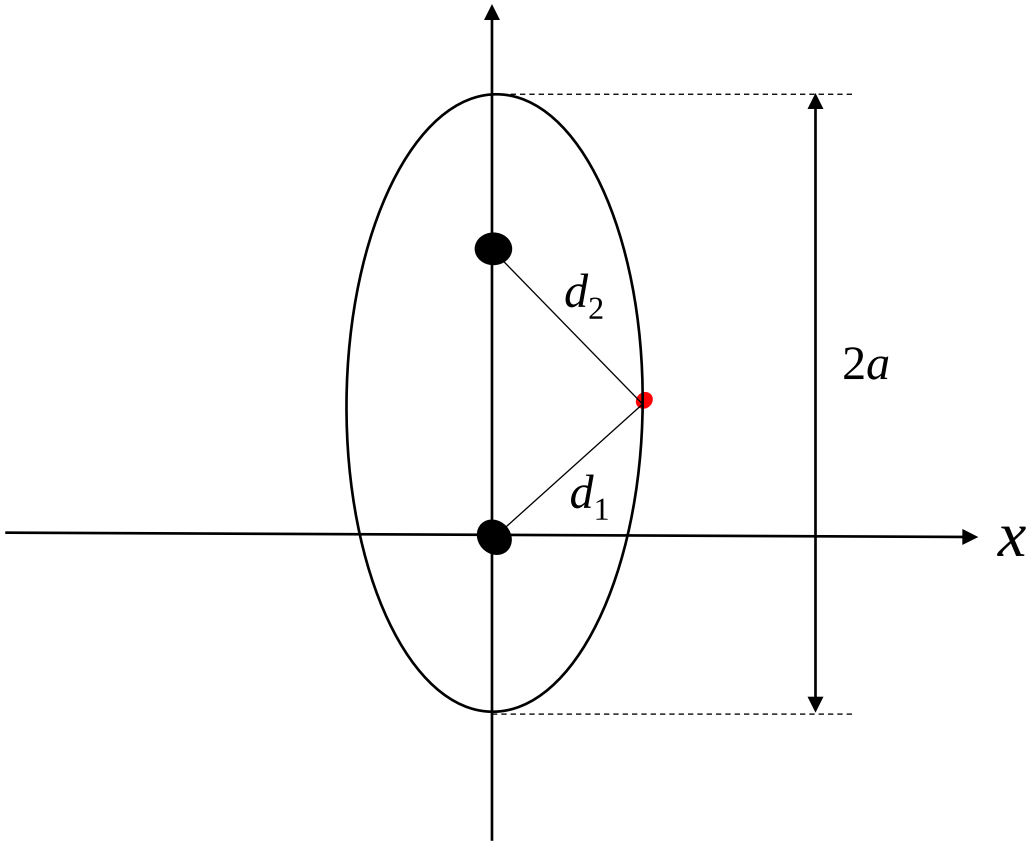


The angle θ can be found from the initial distance, y_i, from the focus that is at the origin of the coordinate system.

Property of an Ellipse

For any point on an ellipse, the sum of its distances to the two foci is equal to the length of the major axis.

$$d_1 + d_2 = 2a$$



Elliptical Orbit Rotation

The distance from (x,y) to the two foci gives

$$x^2 + y^2 = {y_i}^2$$

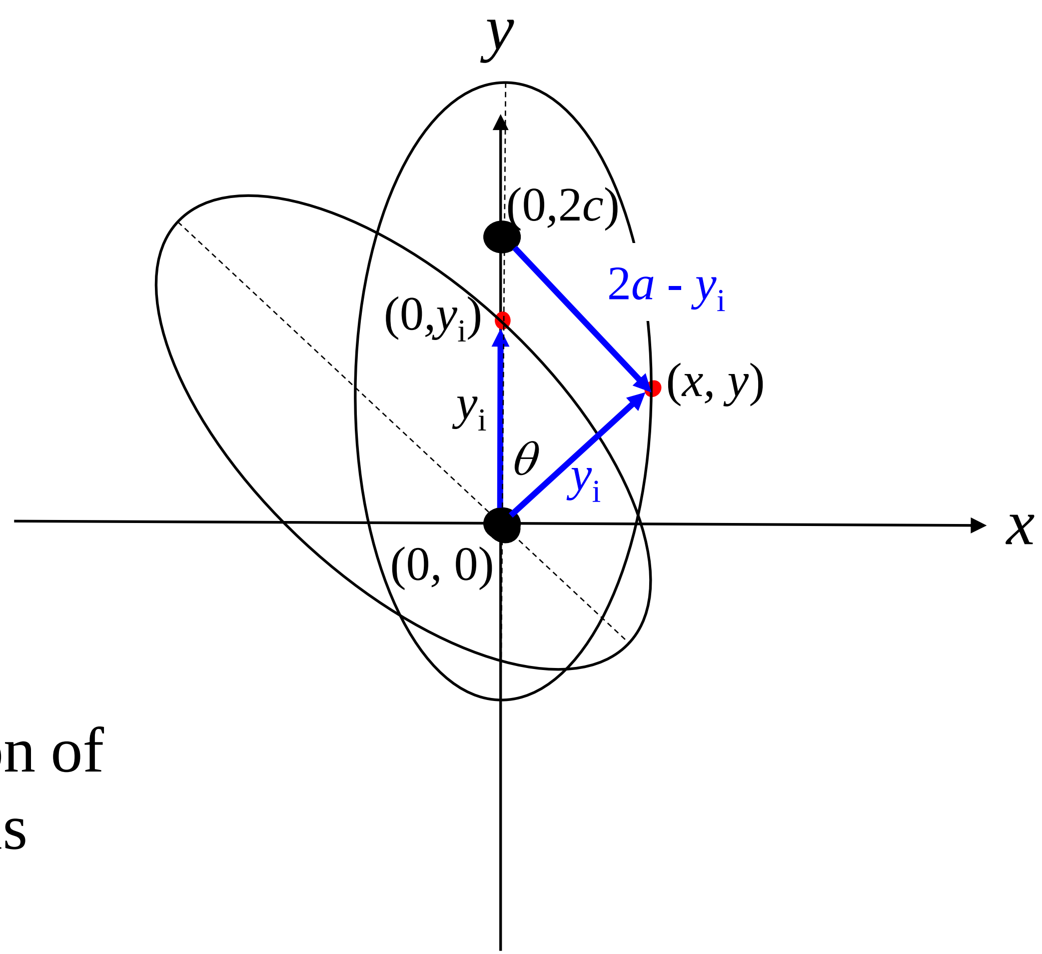
$$x^2 + (y - 2c)^2 = (2a - y_i)^2$$

Solve for y

$$y = \frac{c^2 - a^2 + a y_i}{c} \qquad x = \sqrt{{y_i}^2 - y^2}$$

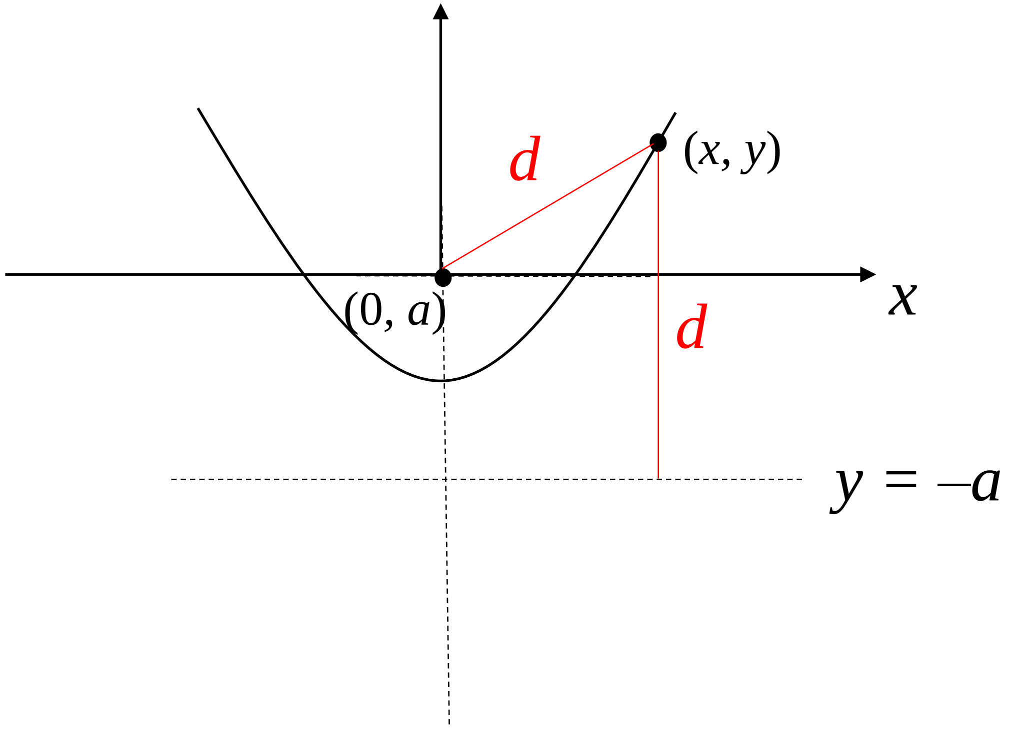
The sign of the rotation depends on the direction of the motion. The sine and cosine of the angle θ is

$$\cos\theta = \frac{y}{\sqrt{x^2 + y^2}} \qquad \sin\theta = \frac{x}{\sqrt{x^2 + y^2}}$$



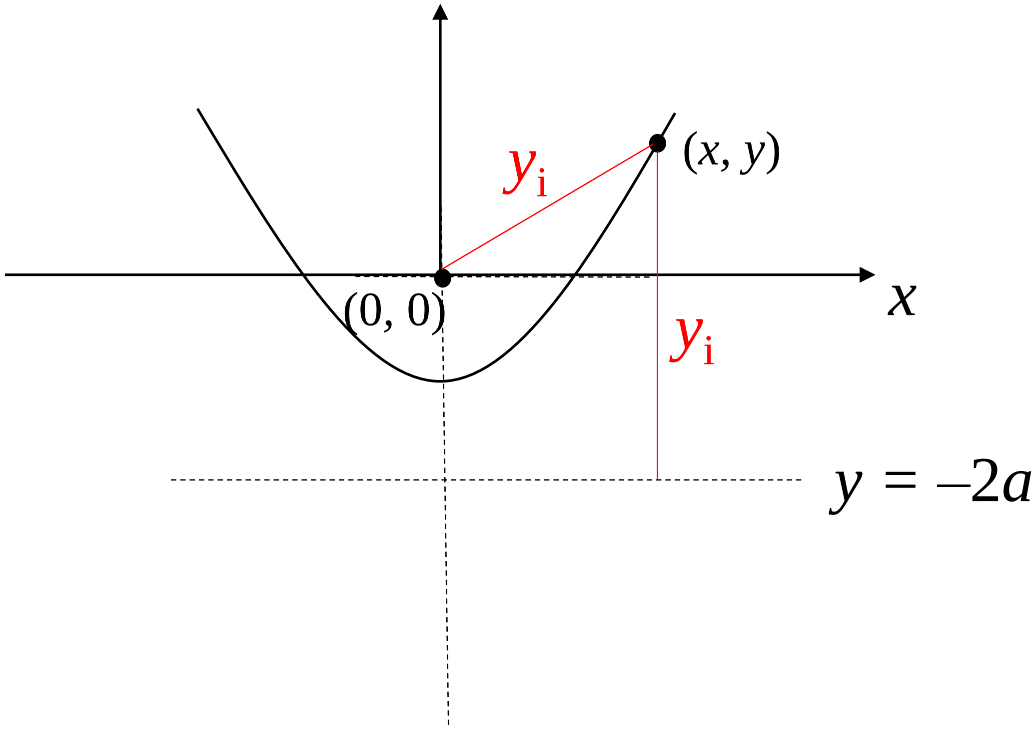
Property of a Parabola

For any point on a parabola, the distance to the focal point is equal to the distance to the directrix.



Parabolic Orbit Rotation

The distance from the point (x, y) to the focal point, which is at the origin, is equal to the distance to the line $y = -2a$.



$$x^2 + y^2 = {y_i}^2$$

$$y + 2a = y_i$$

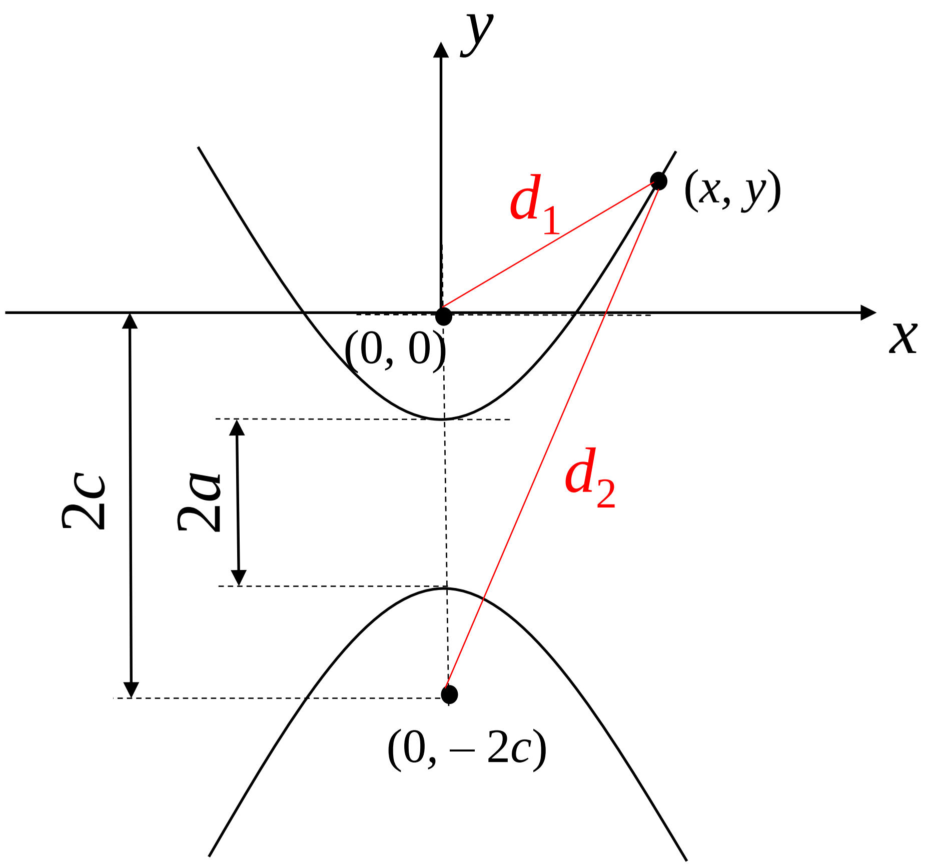
$$y = y_i - 2a$$

$$x = \sqrt{{y_i}^2 - y^2} = \sqrt{4a(y_i - a)}$$

$$\cos\theta = \frac{y}{\sqrt{x^2 + y^2}} \qquad \sin\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

Property of a Hyperbola

$$d_2 - d_1 = 2a$$



Hyperbolic Orbit Rotation

The distance from (x,y) to the two foci gives

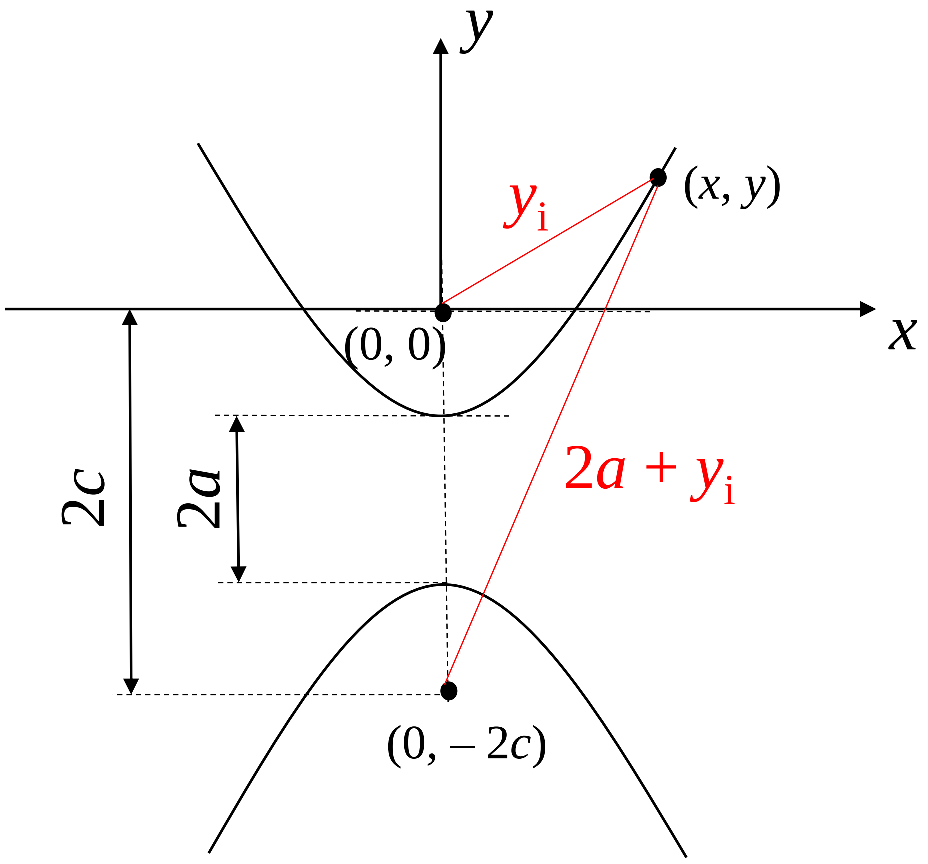
$$x^2 + y^2 = {y_i}^2$$

$$x^2 + (y + 2c)^2 = (2a + y_i)^2$$

Solve for y and x

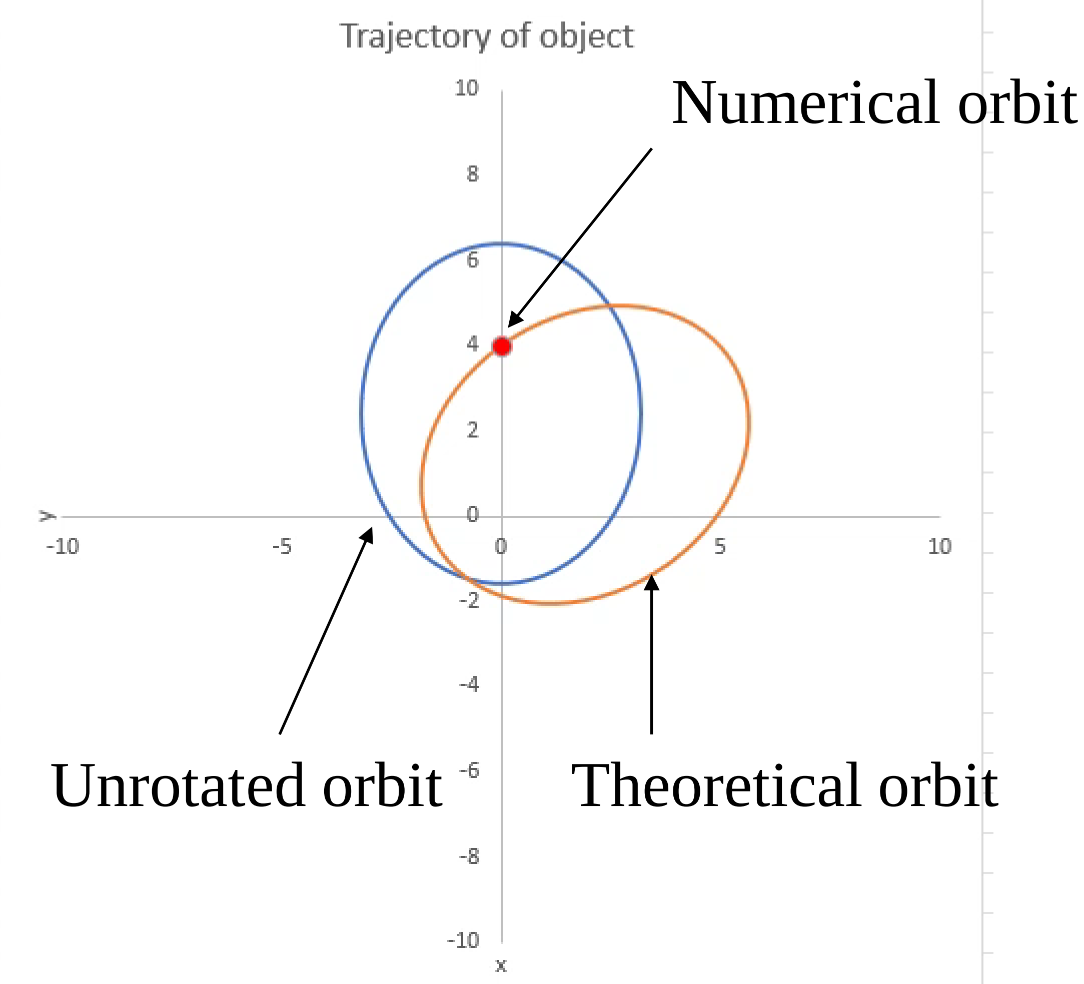
$$y = \frac{a^2 - c^2 + a y_i}{c}$$

$$x = \sqrt{{y_i}^2 - y^2}$$



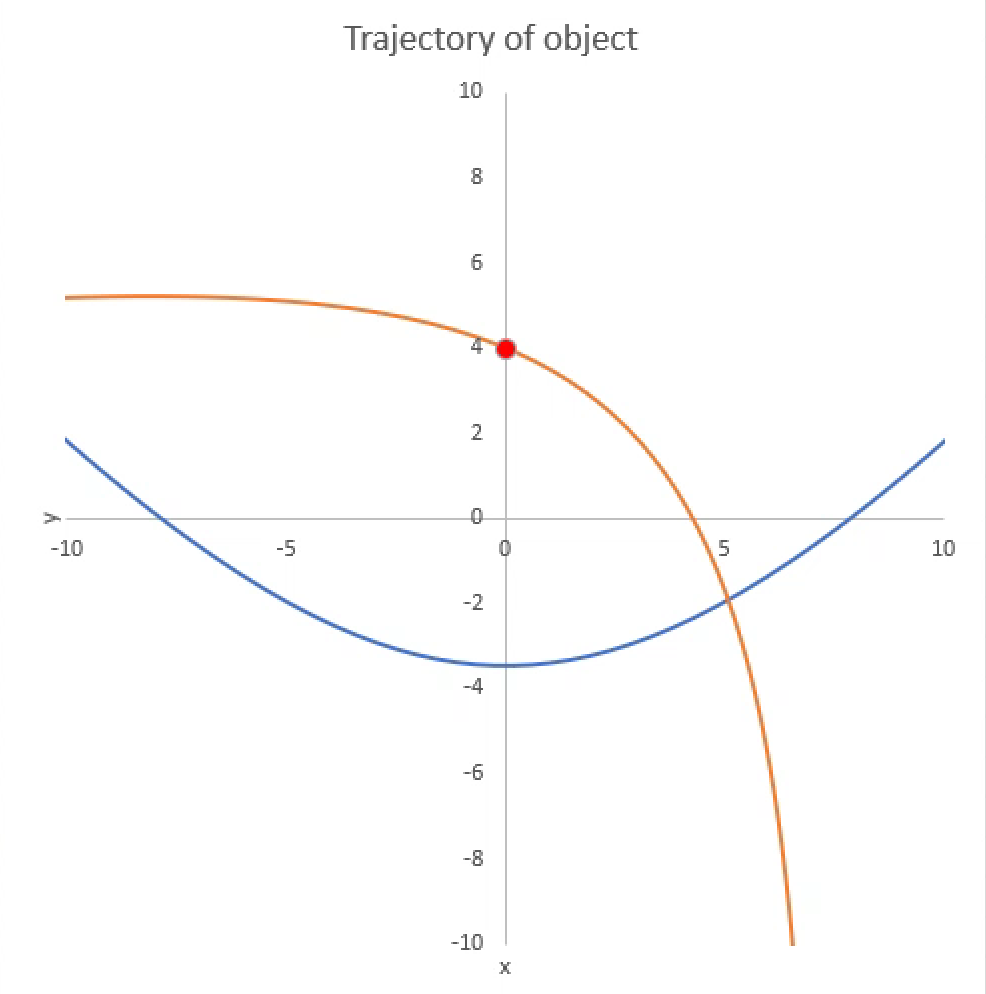
Play movie to see a working program

A	B	C	D	E
Counter	1000	# of steps	1000	
Time step dt	0.0001			
Initial Vx	0.4		yi para	8
Initial Vy	0.3		yi circle	4
Init x (=0 always)	0			
Initial y	4			
Initial E	-0.125			
Initial L	-1.6			
Current Vx	0.4			
Current Vy	0.3			
Current x	0			
Current y	4			
Next Vx	0.4			
Next Vy	0.29999375			
Next x	0.00004			
Next y	4.00003		r+	r-
			1.6	6.4
Current E	-0.125		a	4
Current L	-1.6		c	2.4
Current r	4		b	3.2



Play movie to see a working program

A	B	C	D	E
Counter	1000	# of steps	1000	
Time step dt	0.0001			
Initial Vx	-0.7		yi para	3.448275862
Initial Vy	0.3		yi circle	1.724137931
Init x (=0 always)	0			
Initial y	4			
Initial E	0.04			
Initial L	2.8			
Current Vx	-0.7			
Current Vy	0.3			
Current x	0			
Current y	4			
Next Vx	-0.7			
Next Vy	0.29999375			
Next x	-0.00007			
Next y	4.00003		r+	r-
			3.445219	28.44521872
Current E	0.04		a	12.5
Current L	2.8		c	15.94521872
Current r	4		b	9.899494937



Drawing Orbits

Hyperbolic Orbit

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

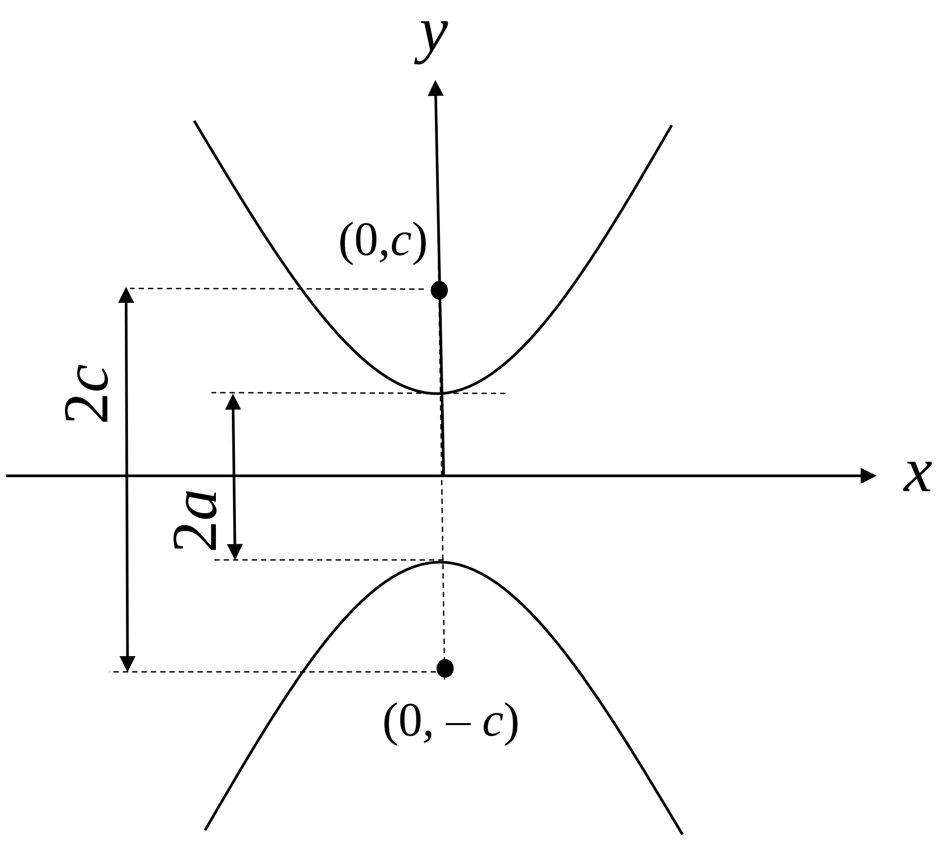
Let $x = b\tan\theta$

$y = a\sec\theta$

Get identity $sec^2\theta - tan^2\theta = 1$

Upper branch $-90° < \theta < 90°$

Lower branch $90° < \theta < 270°$

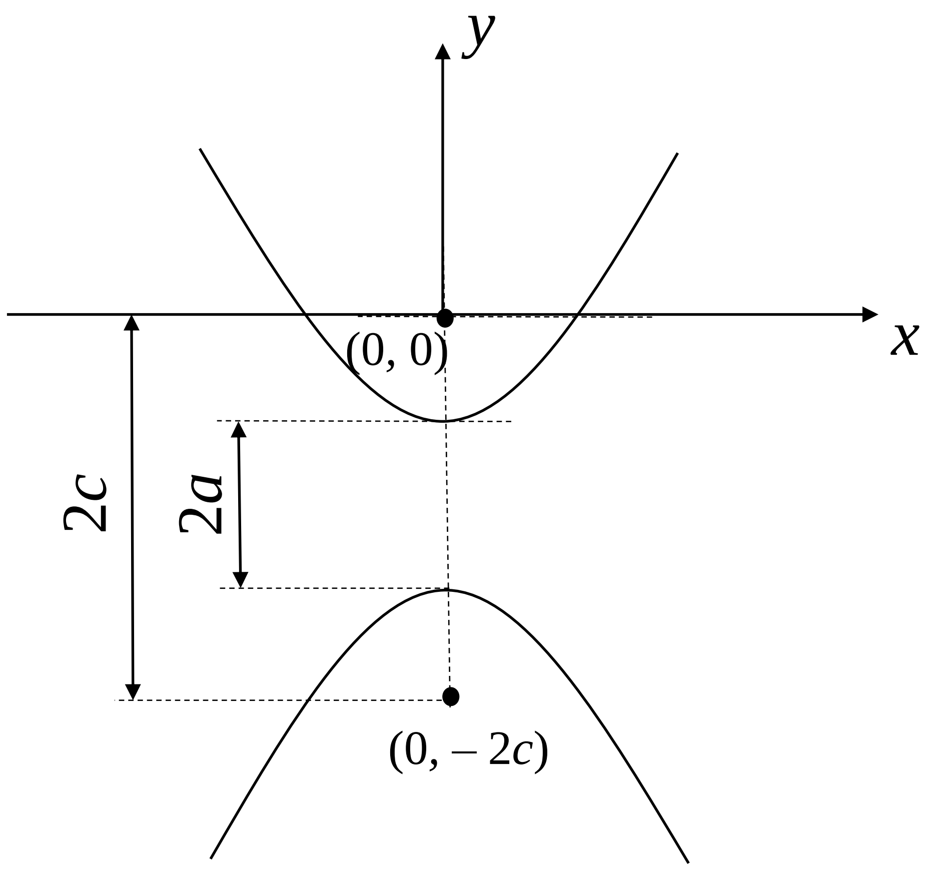


$$a^2 + b^2 = c^2$$

Shifted Origin

$$x = b\tan\theta$$

$$y = a\sec\theta - c$$



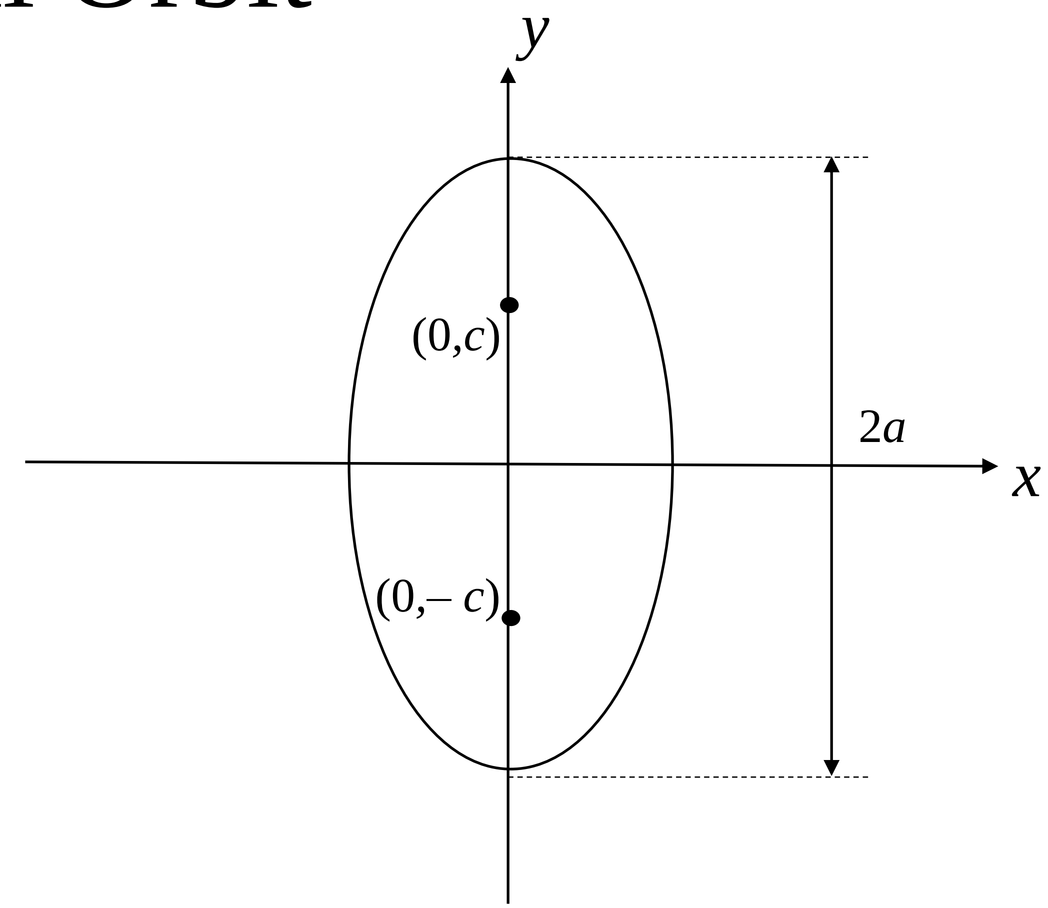
Elliptical Orbit

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

Let $x = b\cos\theta$

$y = a\sin\theta$

Get identity $sin^2\theta + cos^2\theta = 1$

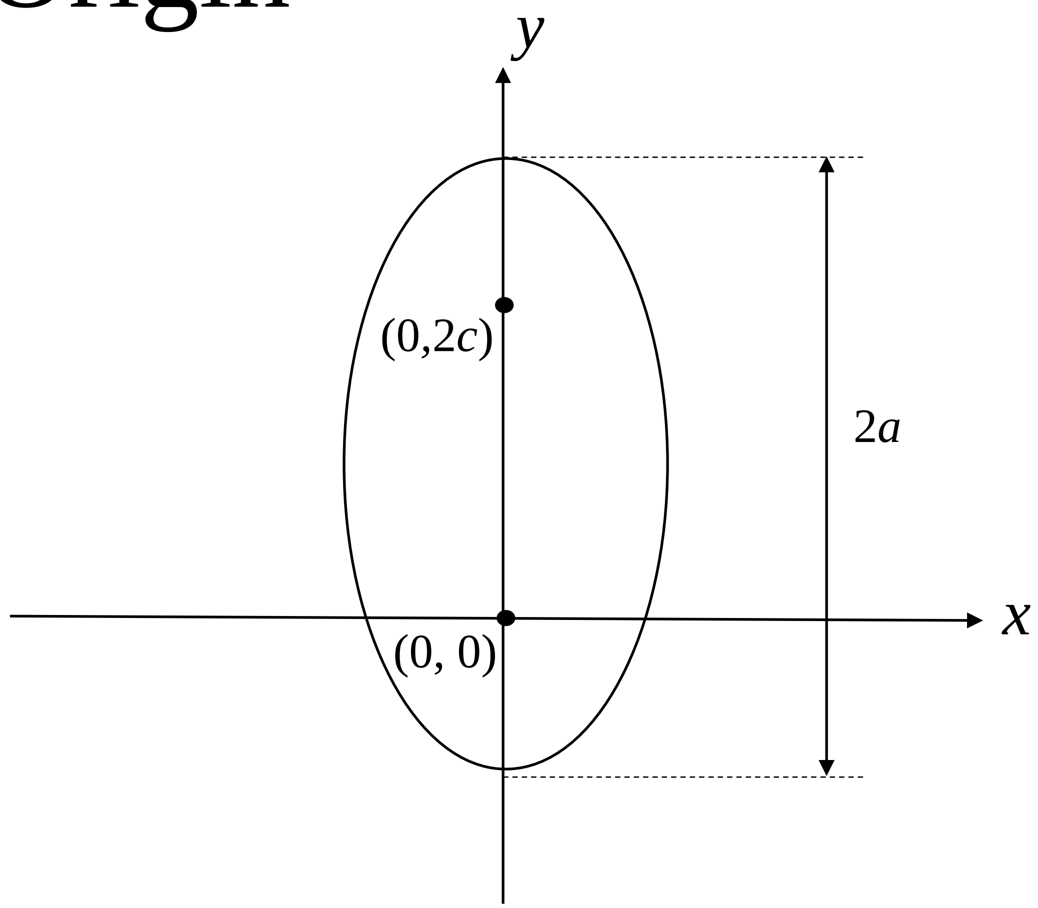


$$a^2 - b^2 = c^2$$

Shifted Origin

$$x = b\cos\theta$$

$$y = a\sin\theta + c$$



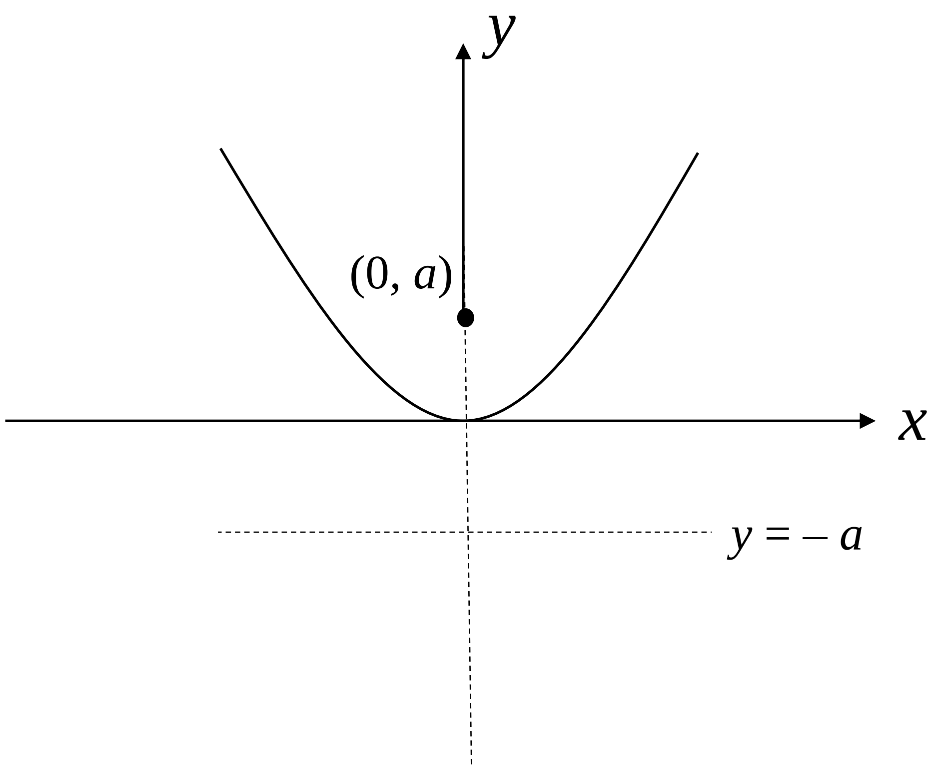
Parabolic Orbit

$$y = \frac{x^2}{4a}$$

Let

$$x = 2at$$

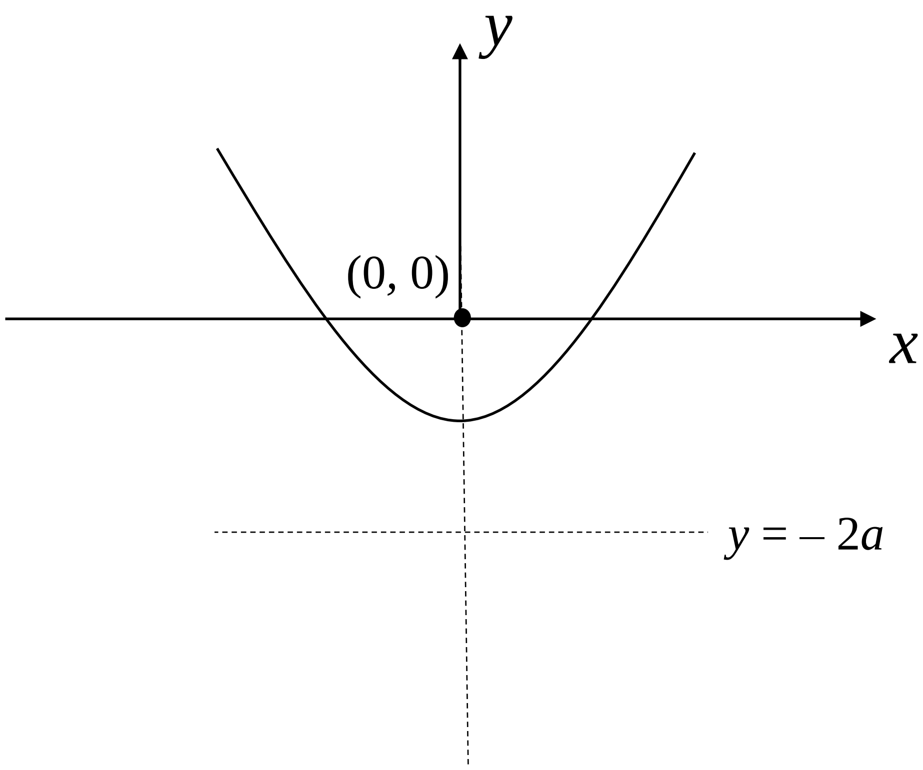
$$y = at^2$$



Shifted Origin

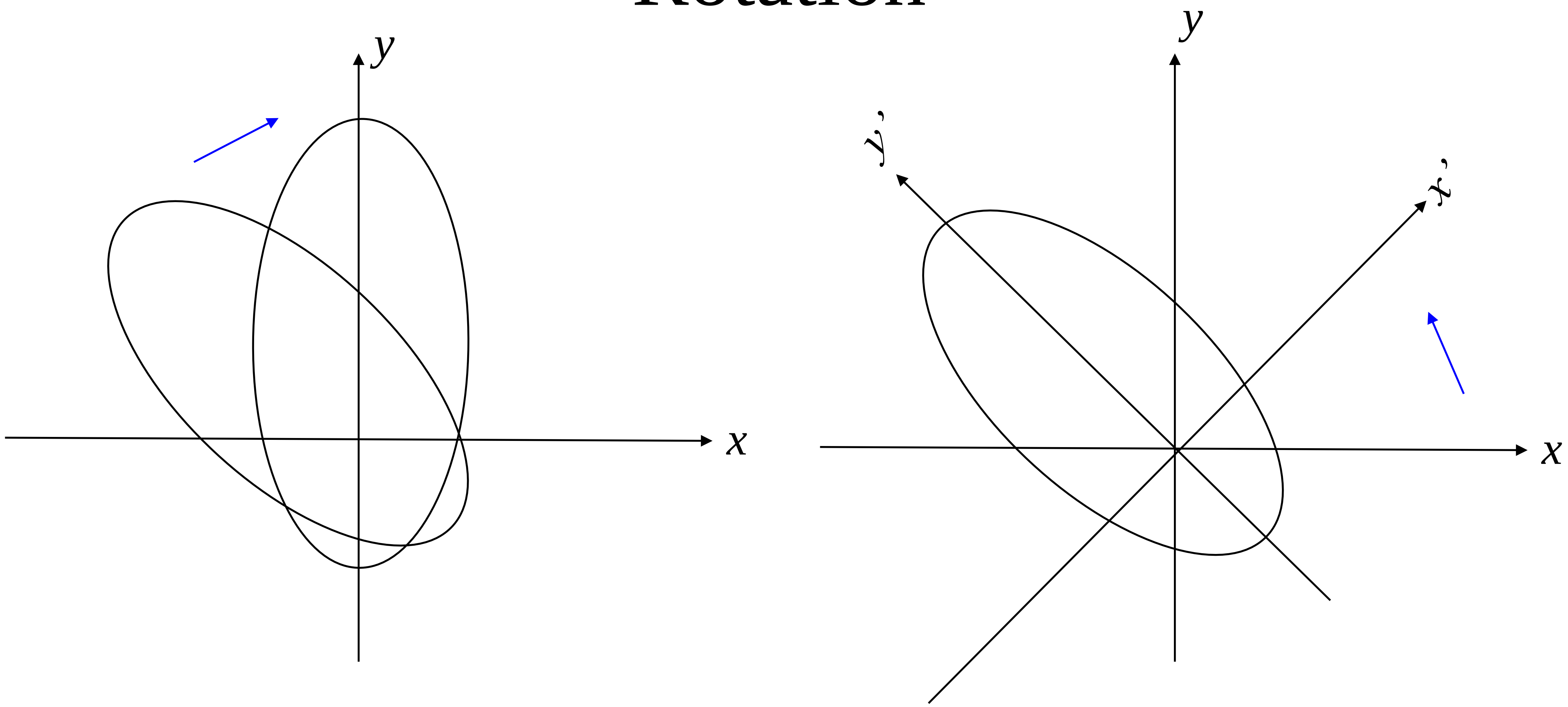
$$x = 2at$$

$$y = at^2 - a$$



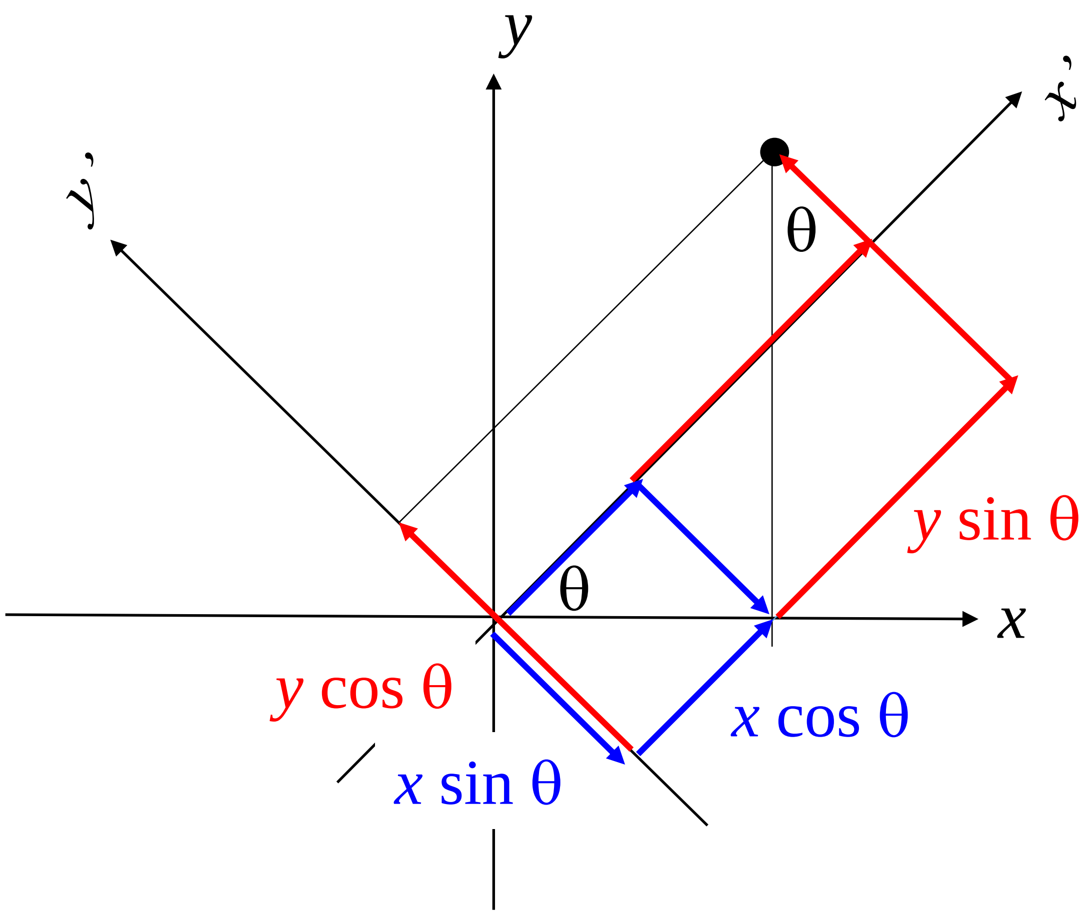
Rotation

Rotation



Rotating a curve clockwise is equivalent to rotating the coordinate axes counter-clockwise. And vice versa.

Counter-Clockwise Axis Rotation



$$x' = x\cos\theta + y\sin\theta$$

$$y' = y\cos\theta - x\sin\theta$$

Clockwise Rotation of Axes

Same as CC rotation with negative angle

$$x' = x\cos\theta + y\sin\theta$$

$$y' = y\cos\theta - x\sin\theta$$

Since the sine function changes sign for negative angle, clockwise rotation with positive angle is

$$x' = x\cos\theta - y\sin\theta$$

$$y' = y\cos\theta + x\sin\theta$$