Fate of a Gambler:

A cautionary tale for cavalier applications of the central limit theorem

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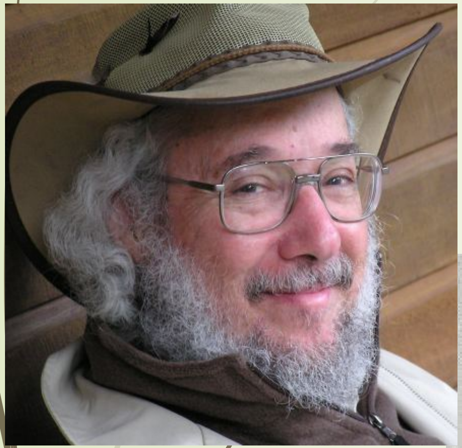

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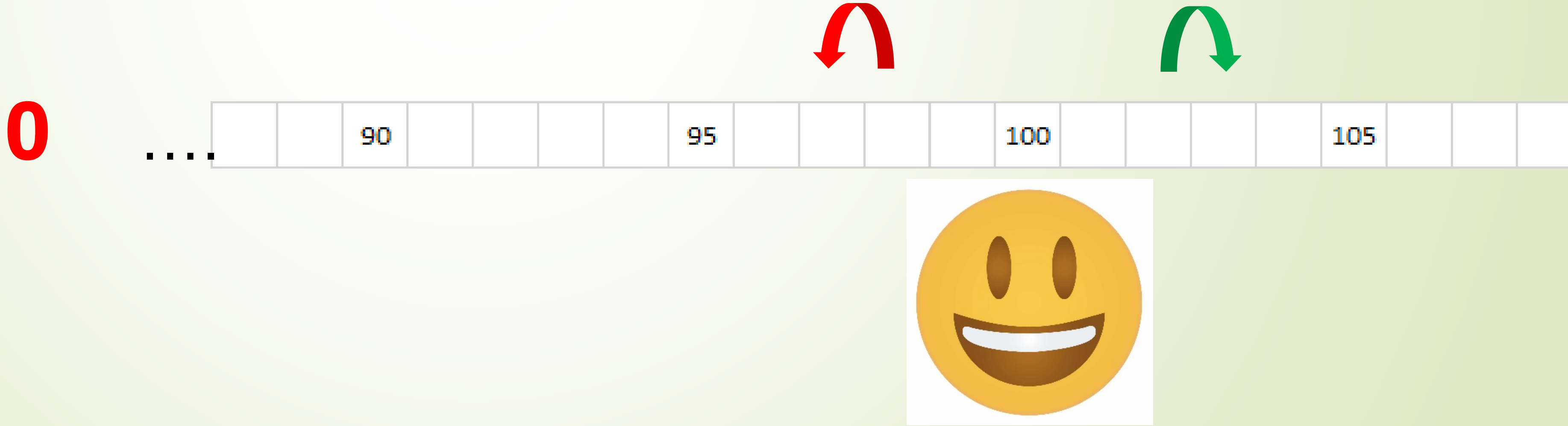
Standard Gambler's Ruin:

A fair coin is tossed. Each time a gambler wins/loses $1 against the house.
Starting with $100, say, he/she will
eventually ***lose it all***.

The earliest known mention of the gambler's ruin problem is a letter from Blaise Pascal to Pierre Fermat in **1656**
https://en.wikipedia.org/wiki/Gambler's_ruin

Standard Gambler’s Ruin:

~ simple Random Walk (unbiased) on a line



A *percentage* version:

A fair coin is tossed. Each time a gambler wins/loses a ***fixed percentage*** (e.g., 10%) of his/her wealth (at the time of the toss) .

He/she will *never* "lose it all", but what can we expect after N tosses?

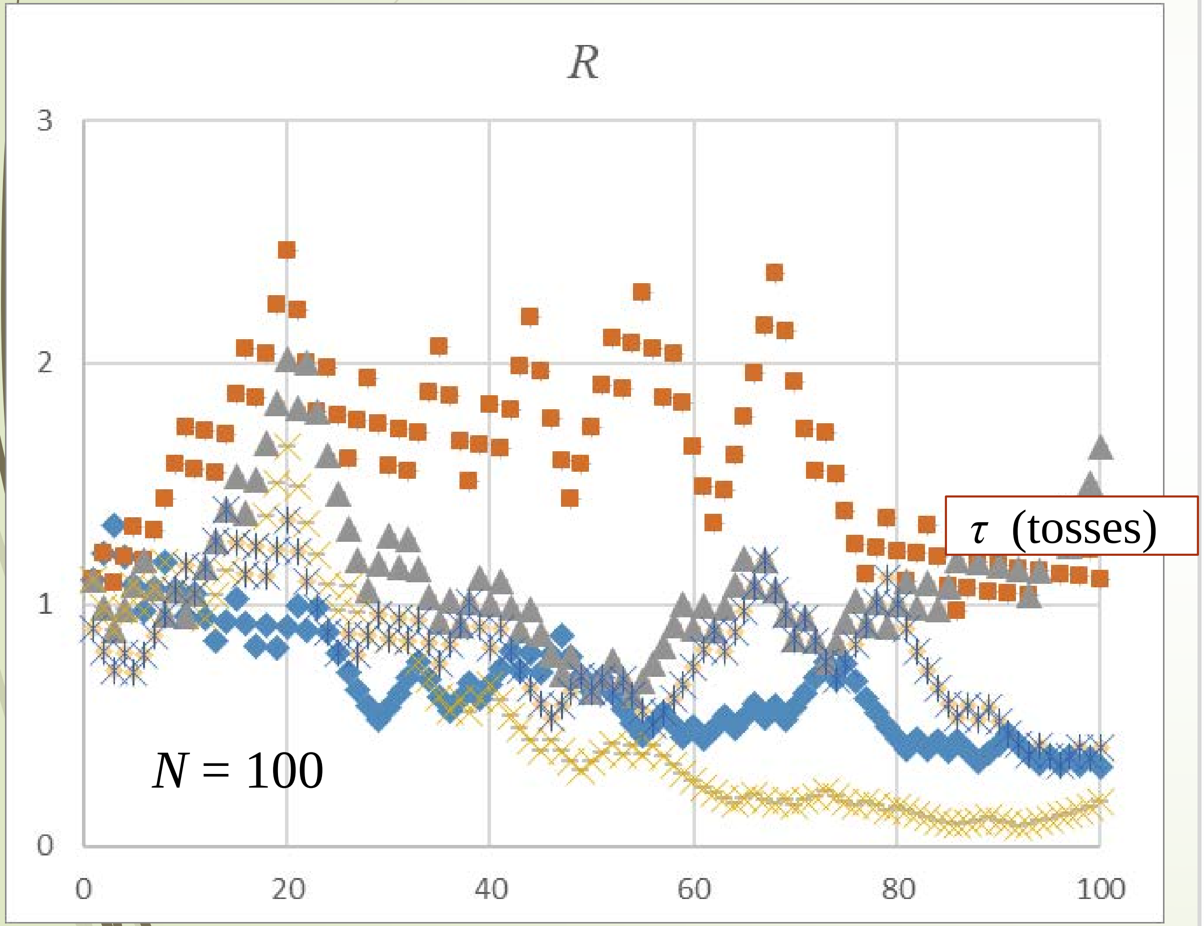
Intuitive guess:

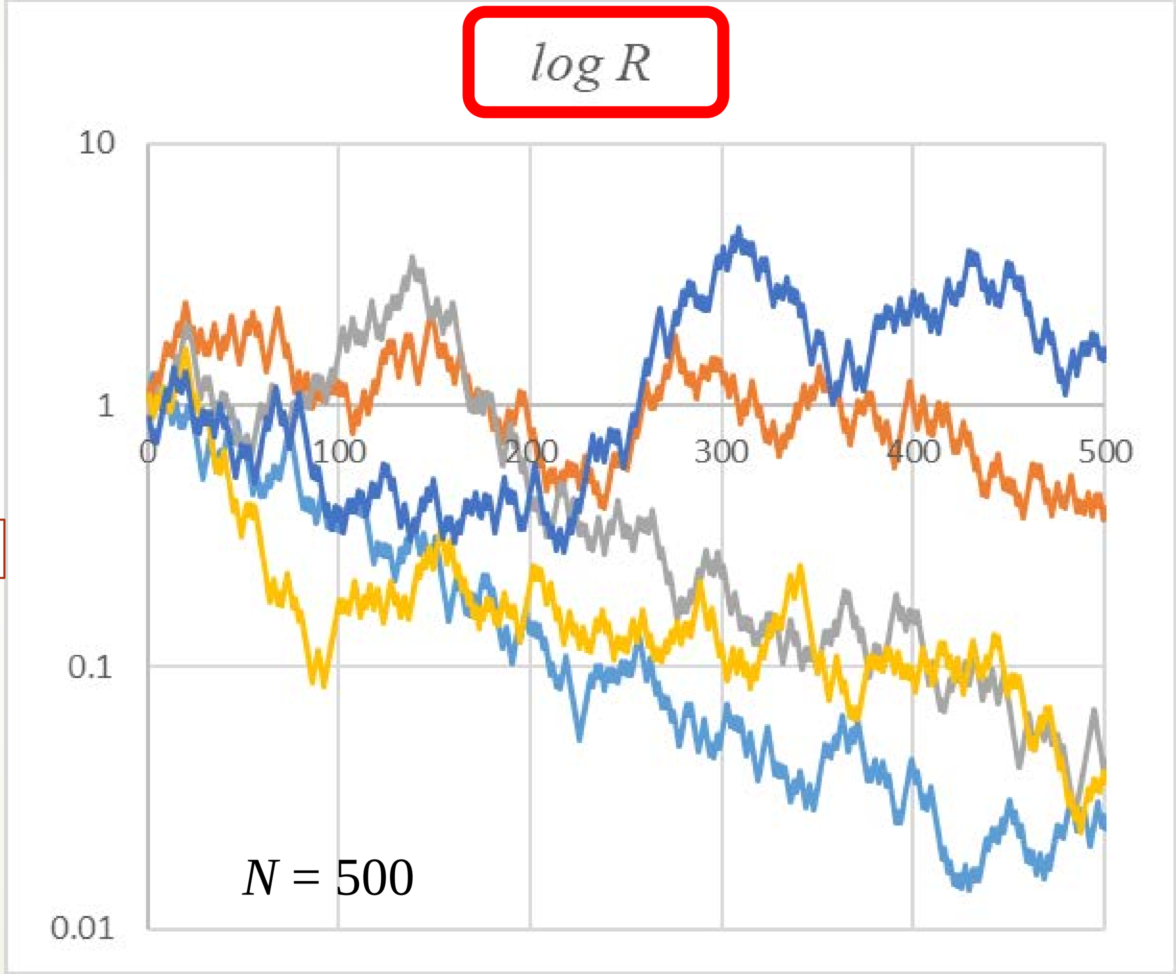
Since the coin is fair, the "return"

$$R \equiv \frac{\text{wealth (at the end)}}{\text{wealth (at start)}}$$

should be **1** (on *average*, for any N)!

Yet, simulations show…





Yet, simulations seem

… to indicate

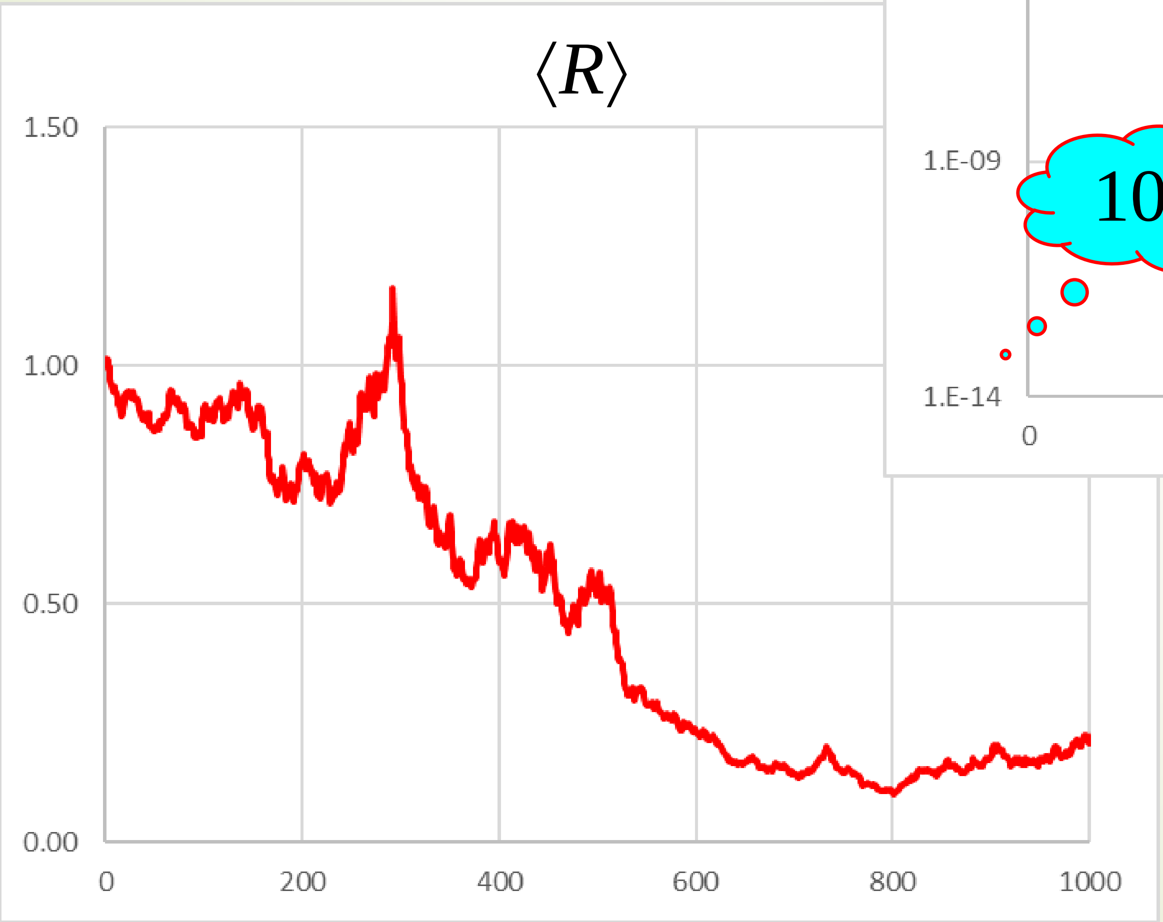
R dropping *precipitously* with N !

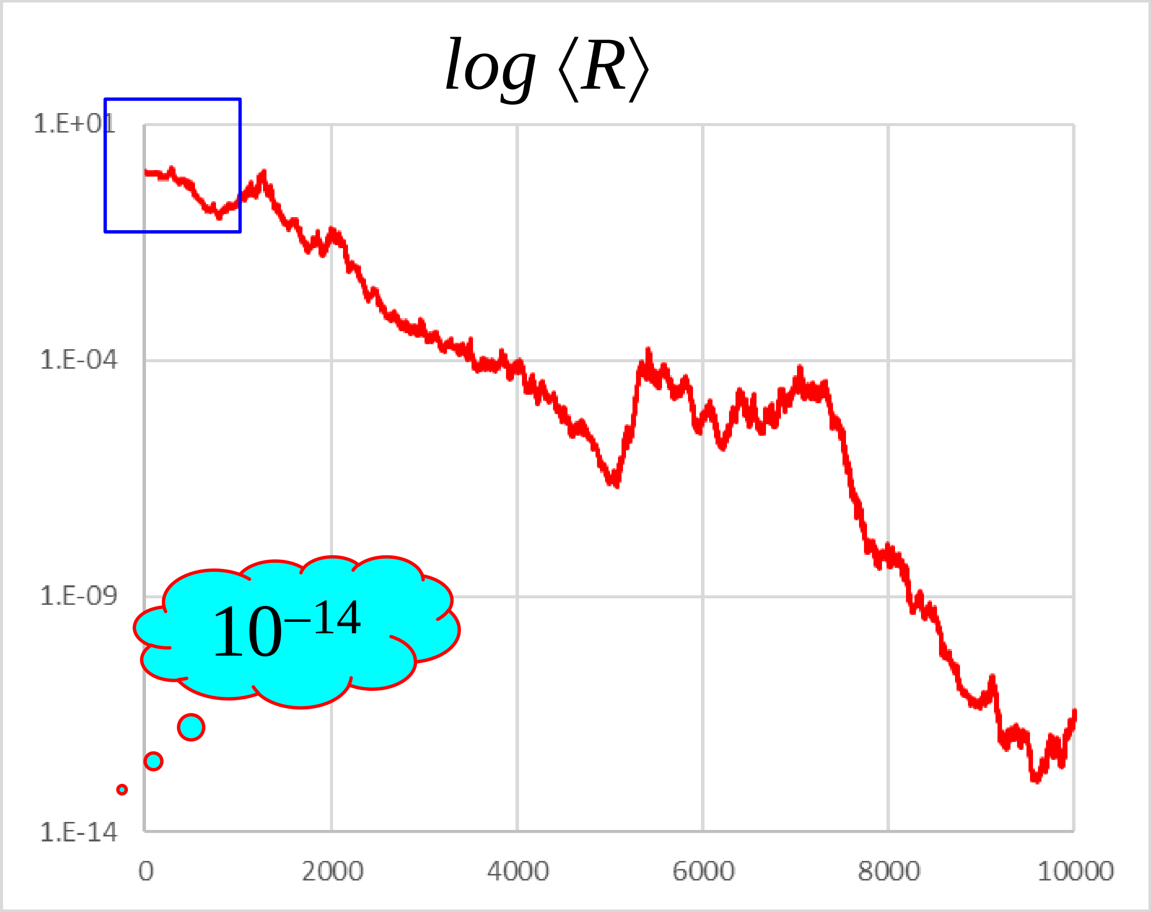
e.g., $\langle R \rangle$ - average over 100 runs,

for N= 1K and 10K:

⟨…⟩ average over 100 runs

$N = 1000$





$N = 10K$

OTOH, if we invoke the Central Limit Theorem,

$\langle R \rangle$ *increases* exponentially with N !

What's the CLT? and
Why should we invoked here?

Intuitive guess is right!

Exact computation shows

$$\langle R \rangle \equiv 1$$

for *any* N !!

Intuitive guess is right!

So, how do both
simulations and **CLT**
lead us astray
… so *badly* !?

For the curious:
It's not a hard calculation.
Ask me at the end if you want.

Here's an initial hint that this game is ***UNFAIR!***

An fair coin means we "win/lose" half the time:

$$W = N/2$$

That's no good enough, since we really need *R*!

BUT, we don't break even (R=1) when we "win" half the time!

$100

$90 $110

$81 $99 $121

lose

win

alf the time:

y need *R*!

1)

alf the time!

3/4 of the time, you ***LOSE***!

YET, if you average over all possible outcomes,

the game is *EVEN* !

–19 –1 –1 +21

$100

lose

$90 $110

win

$81 $99 $121

1/4 *1/2* *1/4*

From one toss to the next,

$$R(\tau+1) = \begin{matrix} [1+s] \\ [1-s] \end{matrix} R(\tau)$$

s is the "stake" e.g.,10%

From one toss to the next,

$$R(\tau+1) = [1+xs]\,R(\tau)$$

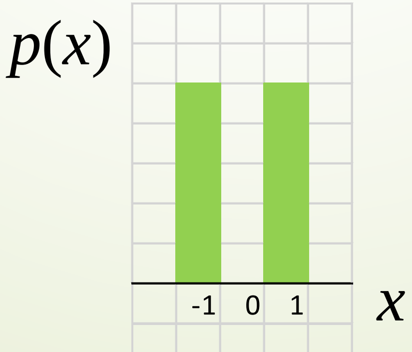
s is the "stake" e.g.,10%

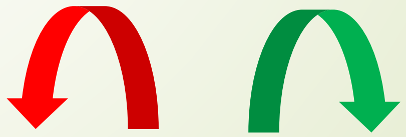
$x = \pm 1$ for win/loss of the toss.

So, a “history” of the game is given by a string of *x*’s:

$$x_1,\ x_2,\ \ldots\ x_\tau\ ,\ \ldots\ x_N$$

just like in the STANDARD game





So, a "history" of the game is given by a string of *x*'s, and at the end,

R is a *product* of these factors:

$$R = [1+x_N s] \ldots [1+x_1 s]$$

To change the ***product*** of a string

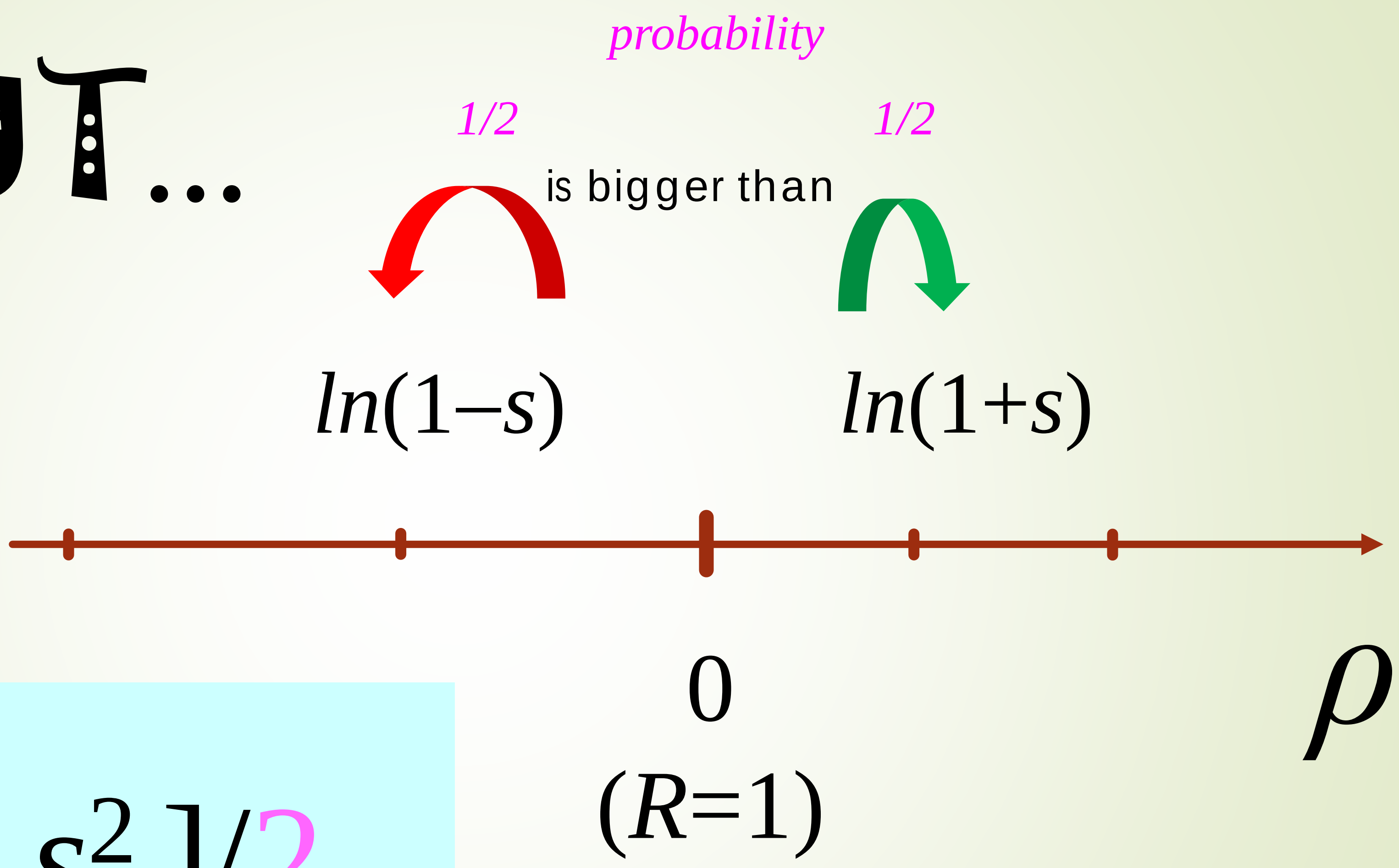
to the ***sum***

(so that we can use the mapping to a random walk and CLT),

just use *logarithm*!!

$$\rho_N \equiv \ln R(N) = \sum_{\tau=1}^{N} \ln\left[1 + x_\tau s\right]$$

BUT…



$$\mu = ln[1 - s^2]/2$$

$$\cong - s^2/2 < 0$$

BIASED Random Walk (in ρ space)!

…with <u>unequal steps</u>

On the average, you ***LOSE*** !

$$\mu = ln\,[1 - s^2\,]/2 \;\cong - s^2/2$$

$$\sigma^2 = -\,[ln(1{-}s)/ln(1{+}s)]^2/4 \;\cong\; s^2$$

To get an idea of how badly off we are with this

BIASED Random Walk,

…let's exploit the CLT

to see what $P(\rho_N)$ is like

(after N tosses).

For the experts:
The exact P is just a fancier binomial. But, getting the fraction of loss is not easy: No closed form for partial sums of binomials! Thus, CLT. Also, for a dice instead, exact P would be impossible!

What's the CLT? and

Suppose x is a random variable picked from some distribution, $p(x)$, with finite mean μ and variance σ^2

st. dev. σ

Generate N of them and ***add***:

$$X \equiv x_1 + \ldots + x_N$$

…and call the induced distribution for X $P(X)$.

For the experts and the curious: $\tilde{P} = [\tilde{p}]^N$

What's the CLT? and

An excellent approximation for P is $\mathcal{N}$, the ***normal*** (Gaussian) distribution, if N is large.

To be specific, $\mathcal{N}(X)$ has
mean $N\mu$ and variance $N\sigma^2$

All you need from p
are its mean and variance!

What's the CLT? and

An excellent approximation for P is $\mathcal{N}$,
the ***normal*** (Gaussian) distribution, if N is large.

s
ce $N\sigma^2$

All other properties of p
are " ***irrelevant*** " !!

All you need from p
are its mean and variance!

i.e., CLT assures us that $P(\rho_N) \cong$

$$\mathcal{N}(\rho_N) \propto exp\left(-\frac{(\rho_N - N\mu)^2}{2N\sigma^2}\right)$$

$\mu \cong -s^2/2$

$\sigma \cong s$

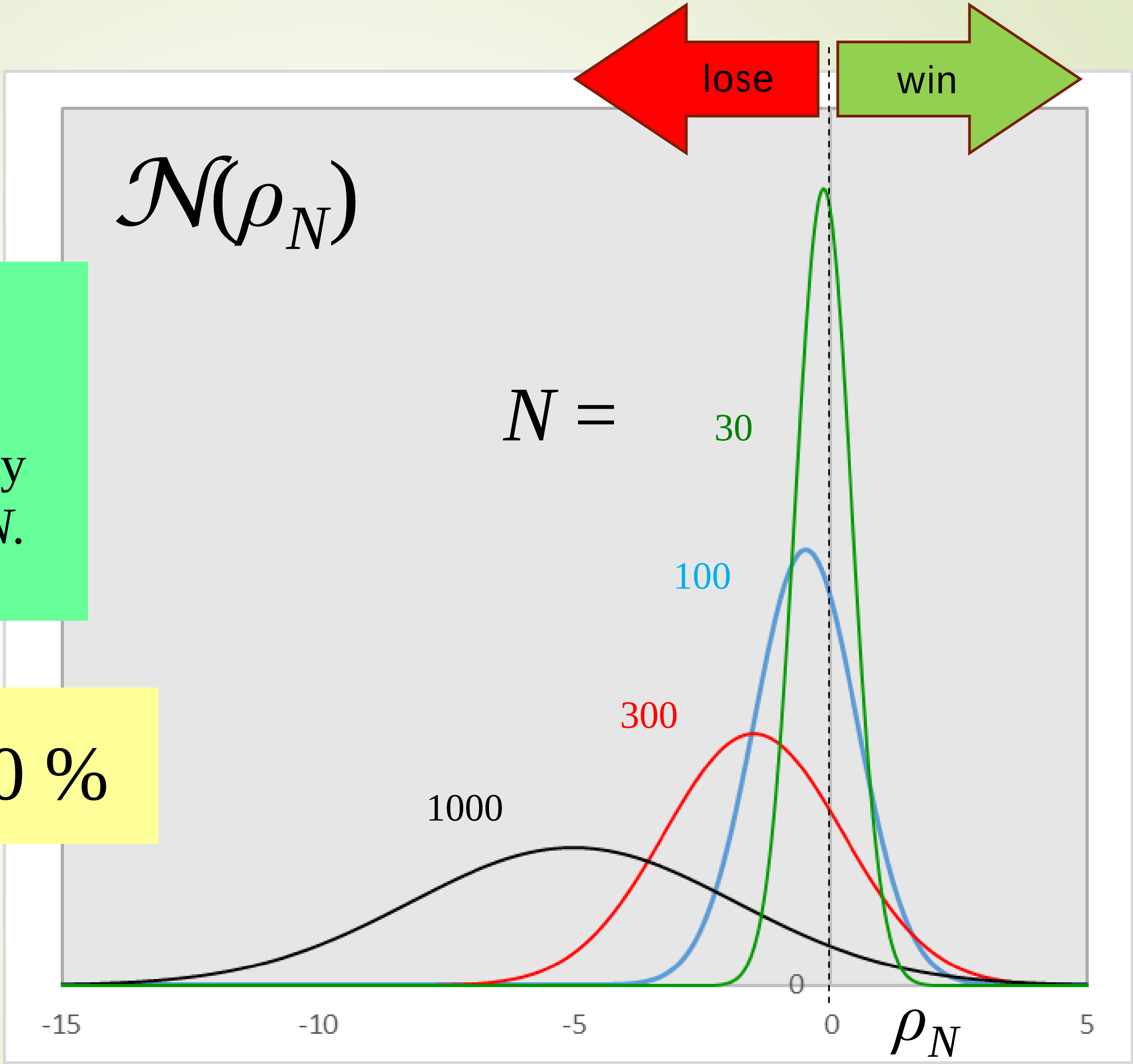
mean $= N\mu$ (*receding* as N !!)

and outpaces st. dev. $\sqrt{N}\sigma$ …

Thus, the fraction of times $\rho < 0$ (lose)
…increases dramatically with N.

$S = 10\ \%$

mean $\cong -Ns^2/2$
st.dev. $\cong \sqrt{N}s$



Thus, the fraction of times $\rho < 0$

…increases dramatically with N.

The CLT has helped us appreciate how simulations gave us the *terrible losses* !

For the experts: Fraction of winning histories ~ $\frac{e^{-N\Phi^2/2}}{\sqrt{2\pi N}\,\Phi}$

But then, the CLT can lead us *ASTRAY*, if we carelessly compute $\langle R \rangle$ with it…

So,

$$\langle R \rangle_{normal} = \langle e^{\rho} \rangle_{normal} =$$

$$\int e^{\rho} \, exp\left\{ - \frac{(\rho - N\mu)^2}{2N\sigma^2} \right\}$$

But then, the CLT can lead us *ASTRAY*, if we carelessly compute $\langle R \rangle$ with it…

can prove this is
POSITIVE!

So,

$$\langle R \rangle_{normal} = \langle e^{\rho} \rangle_{normal} =$$

$$exp\ N\{\mu + \sigma^2/2\}$$

In other words, CLT lead us
to believe that return should…

diverge exponentially with N !!

…an advice that brings
assured ruin.

So, what went so wrong?

(with exploiting the CLT here)

What's the CLT? and Why should we invoked here?

An excellent approximation for P is $\mathcal{N}$, the ***normal*** (Gaussian) distribution, if N is large.

How excellent?

What's "large"?

What's the CLT? and

Uniform

convergence

How excellent?

excellent appro … N,

the ***normal*** (Gaussian) distri … N is large.

If you demand a maximum error of ε ,

you can find an N (that depends on ε) …

so that $\mathcal{N}$ differs from P by less than ε

for ALL X!!

For the experts/purists:
It's actually the CDF, not the PDF, that's uniformly convergent.

What’s the CLT? and

What’s “large”?

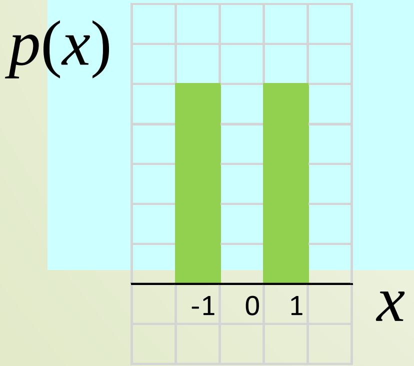
An excellent approximation for *P*

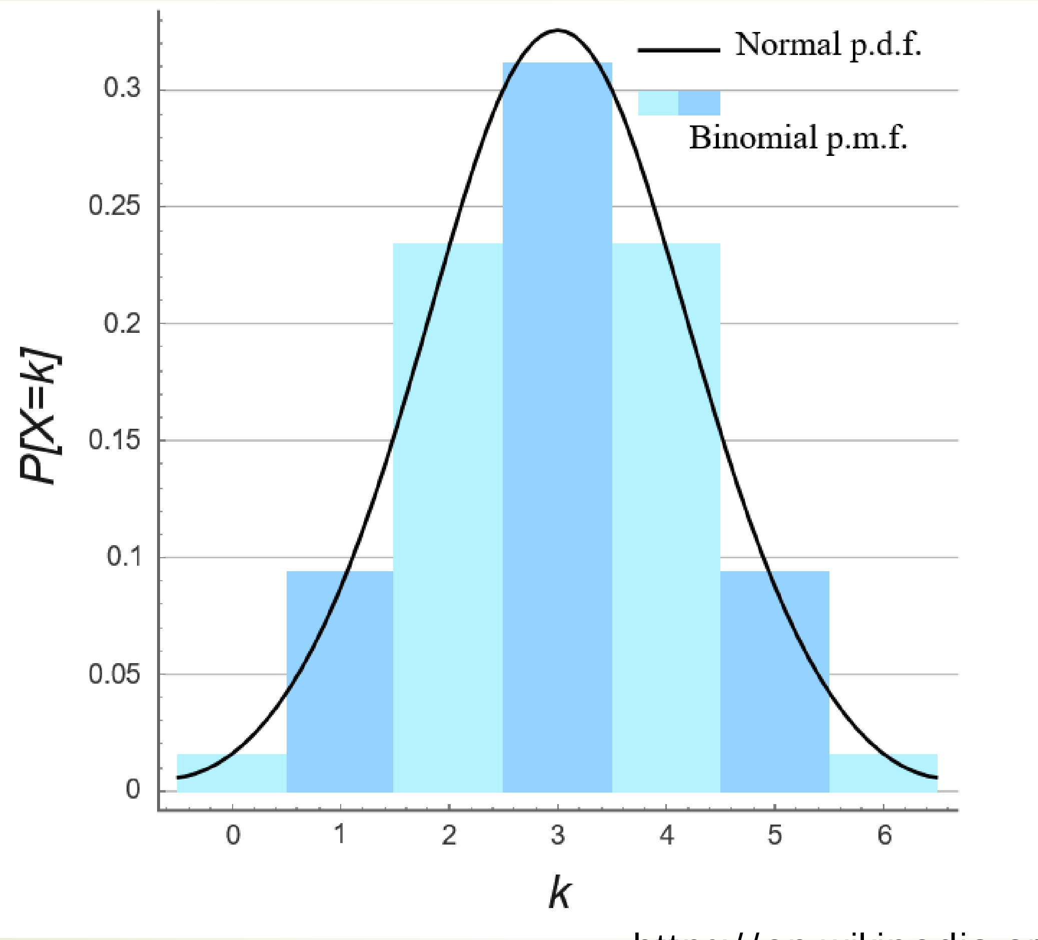
the ***normal*** (Gaussian) distribution, if N is large.

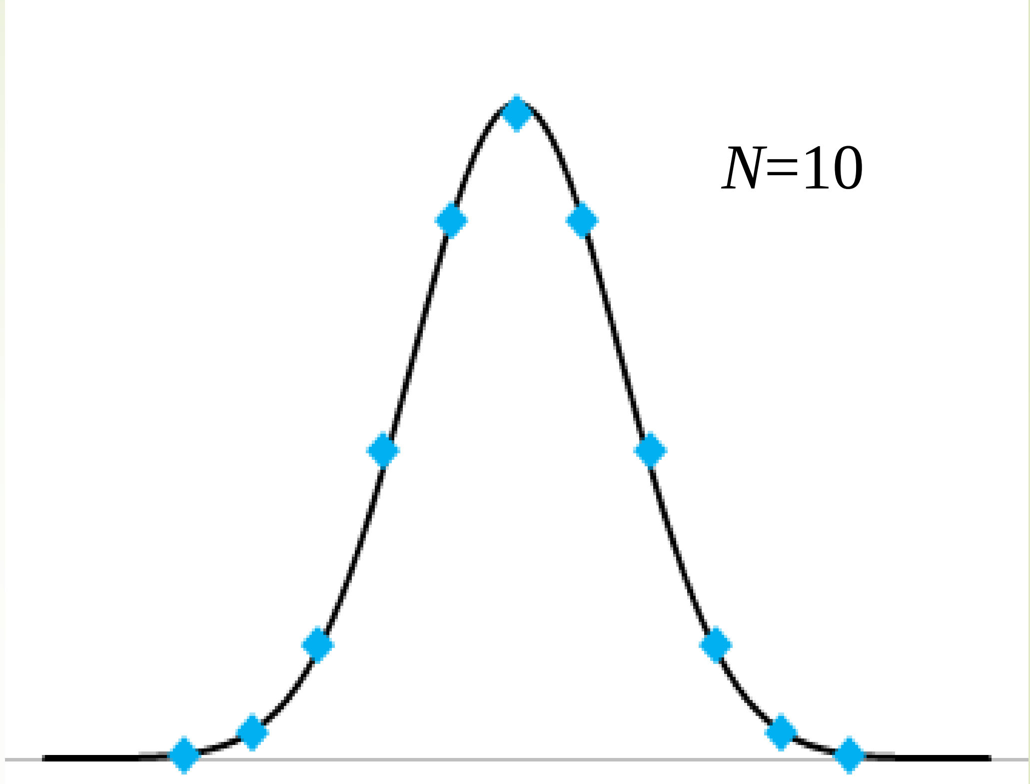
For an example of “how large,” consider the “fair coin” case … labeling $x = \pm 1$ for H/T, so that $\mu = 0$ and $\sigma = 1$.

Meanwhile, $X = -N, -N+2, \ldots, N$

and $P(X)$ is just the binomial.

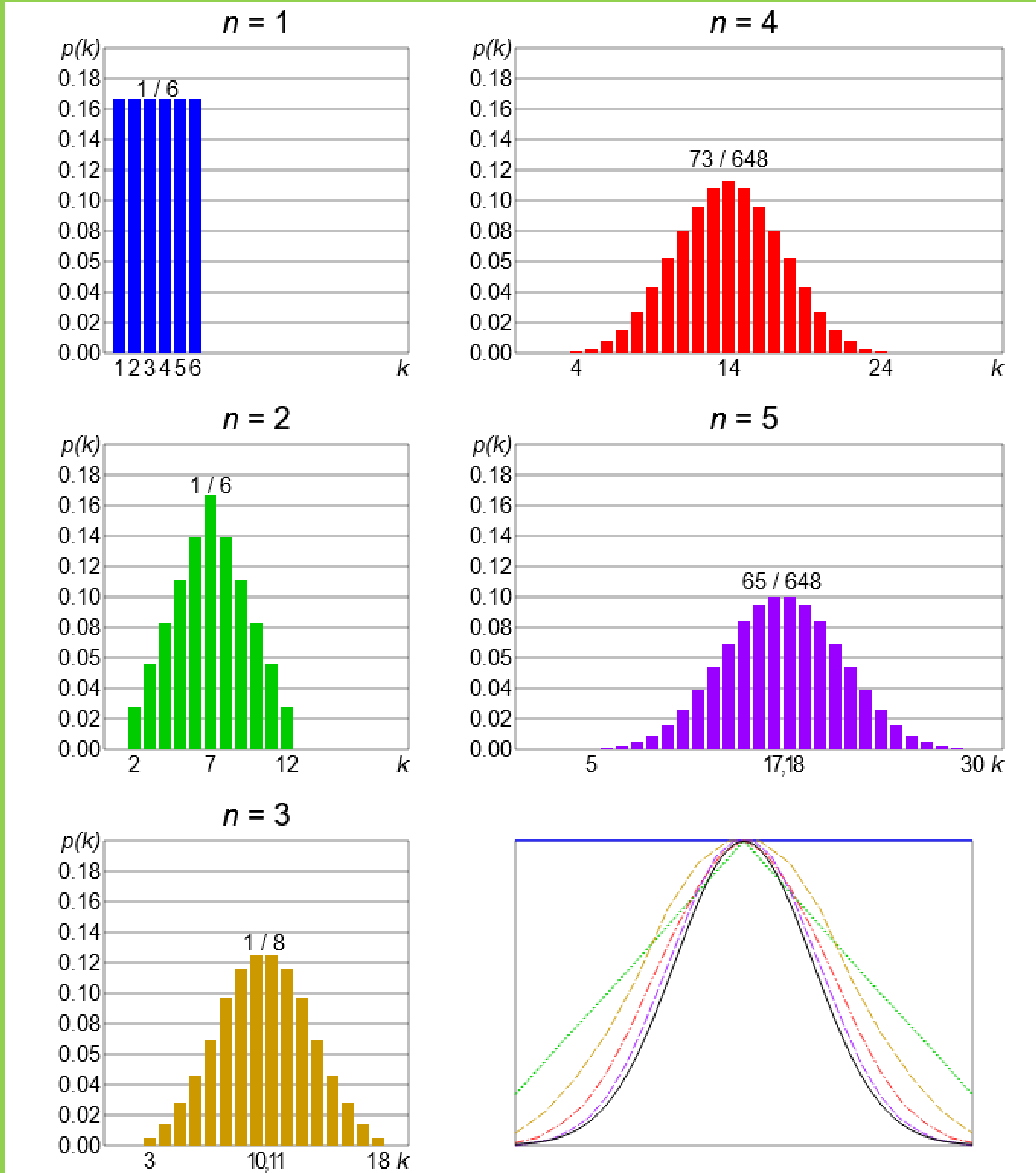






https://en.wikipedia.org/wiki/Binomial_distribution

https://en.wikipedia.org/wiki/Normal_distribution



What's the CLT? and Why should we invoked here?

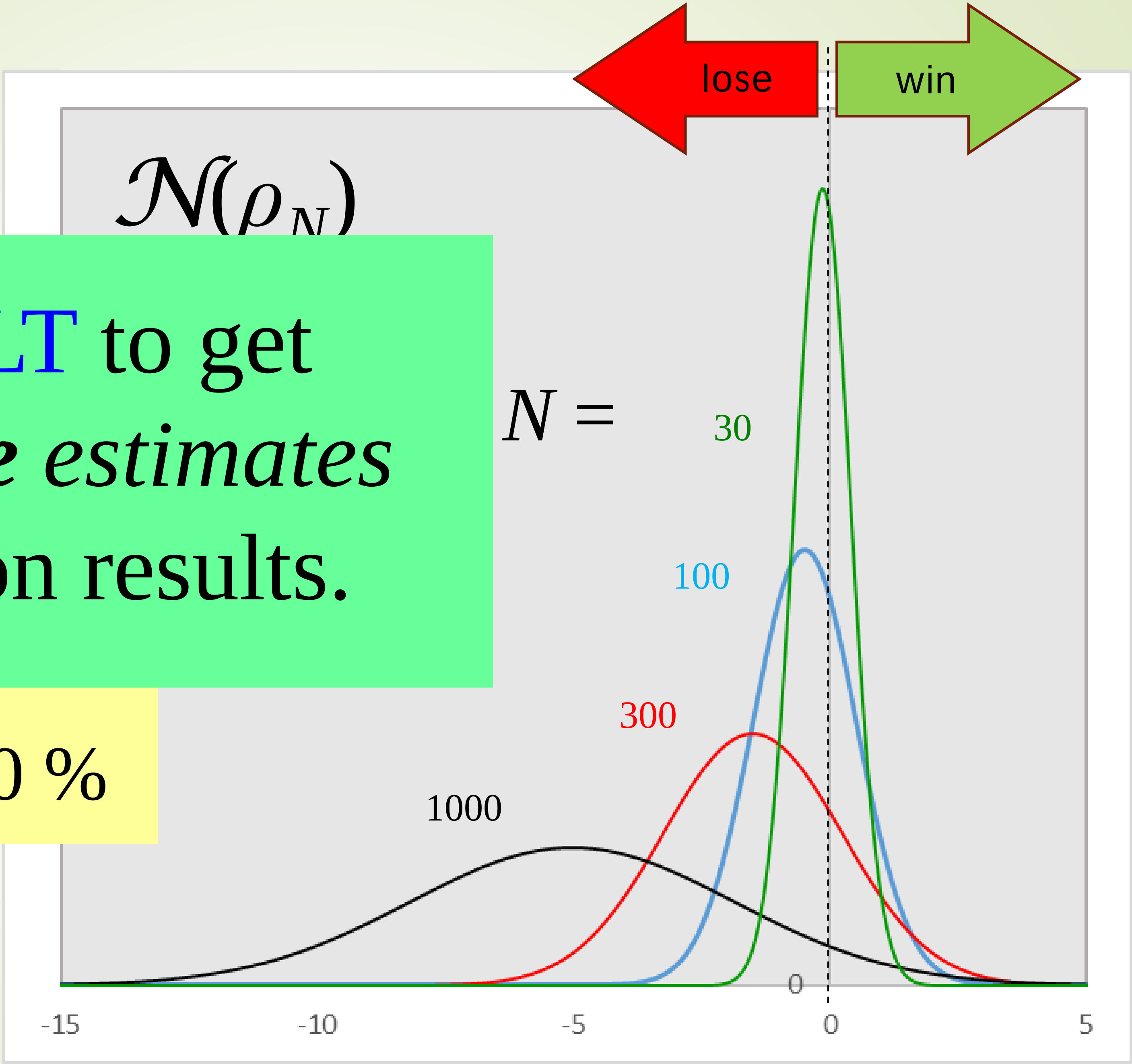
Add N x's of ± 1 and binomial $P(X) \cong \mathcal{N}(X\,;0,N)$

Add N ρ's of $\pm$ whatever and fancy binomial

$$P(\rho_N) \cong \mathcal{N}(\rho_N\,;\,N\mu,\sqrt{N}\sigma\,)$$

All we need are these!

We used the CLT to get *good* ***quantitative*** *estimates* for the simulation results.



$\boldsymbol{S}$ = 10 %

$\mu \cong -Ns^2/2$

$\sigma \cong \sqrt{N}s$

So, what went so wrong? using the CLT to find $\langle R \rangle$?

Though $P(\rho) \cong \mathcal{N}(\rho)$ is excellent for *all* ρ,

$e^{\rho} P(\rho)$ and $e^{\rho} \mathcal{N}(\rho)$

may be quite far apart for some ρ !!

“e^{ρ} has long/large tails !!”

…so that integrals over the above can be quite different.

So, what went so wrong? using the CLT to find $\langle R \rangle$?

Though $P(\rho) \cong \mathcal{N}(\rho)$ is excellent for *all* ρ

!!

For the experts:

Though $\int^X \mathcal{N}(\xi)$ converges *uniformly* to $\int^X P(\xi)$,

$\int^X e^{\xi} \mathcal{N}(\xi)$ does not c.u. to $\int^X e^{\xi} P(\xi)$!

…so that integrals over the above can be quite different.

Take-home messages:

What the CLT tells us is "impressive"!   <u>**BUT**</u>,
… if you want to find the average of anything (associated with *P*),
then you'd better look at the tails of that anything (before using the CLT blindly).

For the experts:
If you want to find the average of some function, f, of the "macroscopic variable" X, then …

- If your f is a function of $\xi \equiv X/N$ *<u>alone</u>*, then, the CLT assures you that $\langle f(\xi) \rangle \to f(\mu)$ for $N \to \infty$.
- If your f is a function of $\xi \equiv X/\sqrt{N}\sigma$ *<u>alone</u>*, AND you know μ=0, then, the CLT assures you that you can use the standard normal distribution to get $\langle f(\xi) \rangle$ for $N \to \infty$.
- In all other cases, study the tails of f *<u>carefully</u>* before relying on the CLT for $\langle f \rangle$!

For the aficionados:
ALL cumulants (beyond the first two, predicted by the CLT) are "***infinitely wrong***"!

Conclusion

If you teach/use the Central Limit Theorem,

please consider an extra warning label:

Do NOT blindly compute ⟨•⟩

with the Gaussian approximant!

Consider carefully …

…the tails of the • first.

Fate of a gambler:

A cautionary tale for cavalier applications of the central limit theorem.

Tossing a fair coin N times, a gambler wins/loses 10% of his/her holdings against the house if each toss is head/tail (H/T). Measuring his/her fortunes by R (ratio of final to initial wealth), then we may ask for $\langle R \rangle$ (the average over all possible 2^N histories). Since the game sounds like it's even, we may guess $\langle R \rangle=1$. When computed exactly, it is indeed so. Yet, when simulations are done, $\langle R \rangle$ drops exponentially with N, e.g., to $O(10^{-14})$ for $N=10^4$. A further puzzle is the following: It is tempting to apply the central limit theorem and replace the distribution of $H-T$ by a normal (since the exact one is just a binomial, in lnR). Replying on that leads to an $\langle R \rangle$ that *increases* exponentially with N! Along with resolutions to these paradoxes, I propose that we add a "warning label" when the central limit theorem is taught.