Projectile Motion:

More Geometry, More Physics

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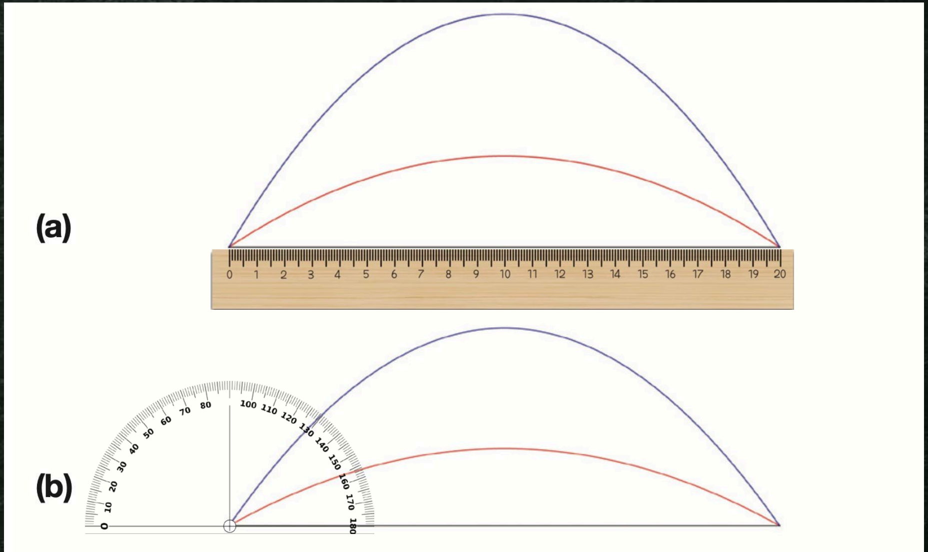
Projectile Motion: More Geometry, More Physics
CSAAPT Fall meeting, October 19, 2024

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0\Delta t + \frac{1}{2}\vec{g}\Delta t^2$$



Which pair of trajectories properly depicts projectile motion?

$$16h_1h_2 = R^2$$

h …height

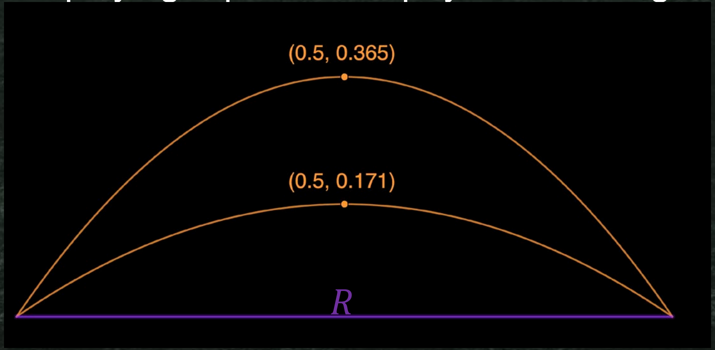
R …range

$$16h_1h_2 = 16 \times$$

Unitless quantities?

Scaling

Scaling to dimensionless form: In some problems, scaling is used to make quantities dimensionless by normalizing them to some characteristic value. This is common in simplifying equations in physics and engineering.



characteristic value … range R

$$\left(\frac{\Delta x}{R}, \frac{\Delta y}{R}\right) = \left(\frac{R/2}{R}, \frac{h_1}{R}\right) = (0.5, 0.365)$$

$R \to R{=}1$ relative range

$h \to h$ relative height

$$\frac{16 h_1 h_2}{R^2} = 1 = \frac{16 h_1 h_2}{R^2}$$

scaled vs. not scaled quantities

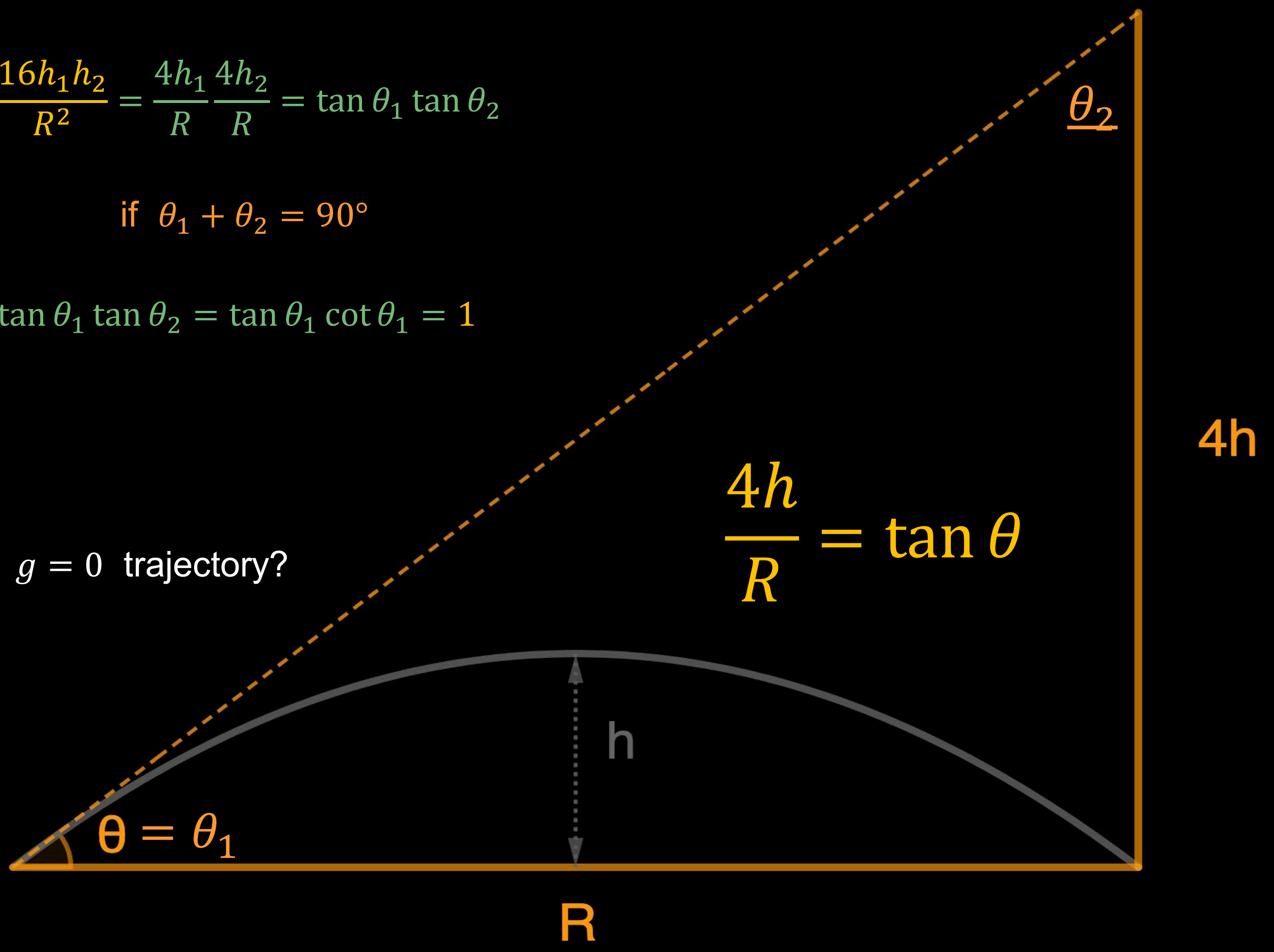
Notice: h_1, h_2 are unique values once a characteristic value is chosen, for example $R = 1$ vs. h_1, h_2, and R have values depend on the units used and the scale of the representation

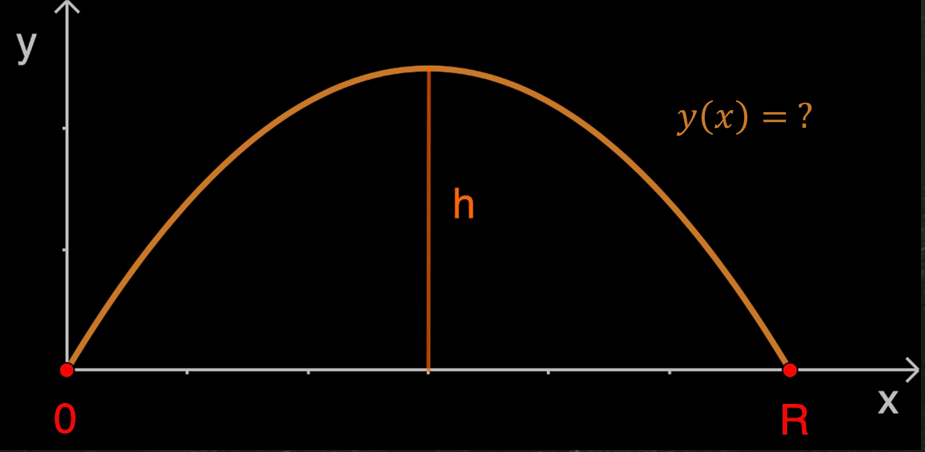
$$\frac{16h_1h_2}{R^2} = \frac{4h_1}{R}\frac{4h_2}{R} = \tan\theta_1 \tan\theta_2$$

if $\theta_1 + \theta_2 = 90°$

$$\tan\theta_1 \tan\theta_2 = \tan\theta_1 \cot\theta_1 = 1$$

$g = 0$ trajectory?





Need two zeros and height h in the middle.

Educated guess: $y = a(x-b)(x-c)$

$b = 0$ and $c = R$

$h = a(R/2)(R/2 - R)$

$y = ax^2 + bx + c$

$y = a(x-b)(x-c)$

$y = a(x-h)^2 + k$

$$y(x) = \frac{4h}{R^2}x(R-x)$$

units ✓

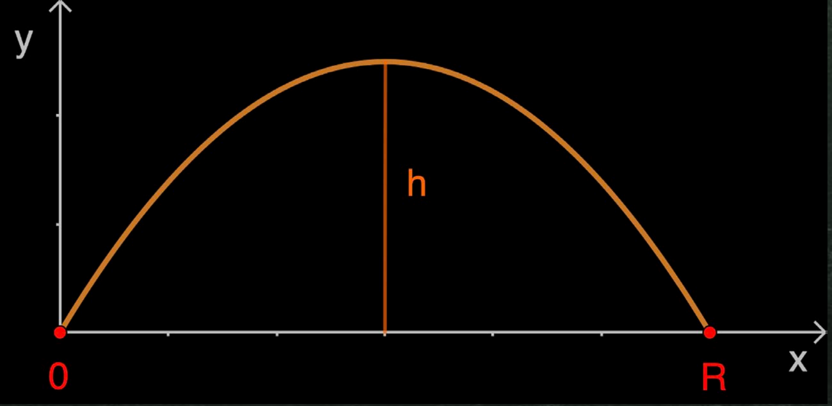
zeros ✓

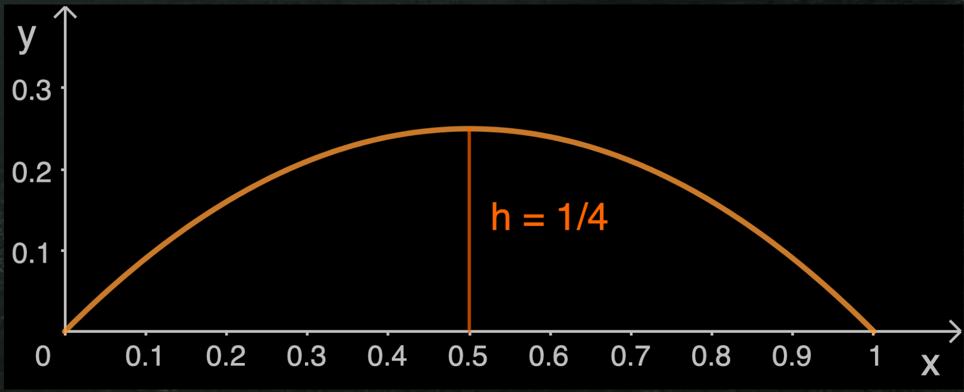
height ✓

when $x \ll R \rightarrow y(x) \approx \frac{4h}{R}x$ … the tangent line to the parabola at the origin

Notice the choice of coordinates and the form of quadratic function.

Factored form $y = a(x-b)(x-c)$ vs. standard form $y = ax^2 + bx + c$.



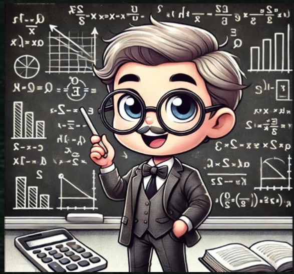


$$y = \frac{4h}{R^2} x(R - x)$$

$$y = x(1 - x)$$

characteristic length R: $\frac{x}{R} \rightarrow x$

characteristic height 4h: $\frac{y}{4h} \rightarrow y$



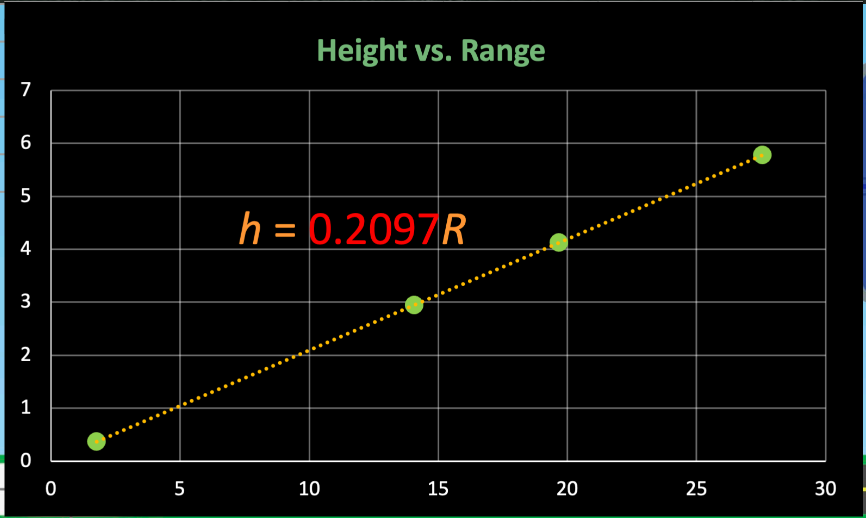
Additional projects using Projectile Motion simulation

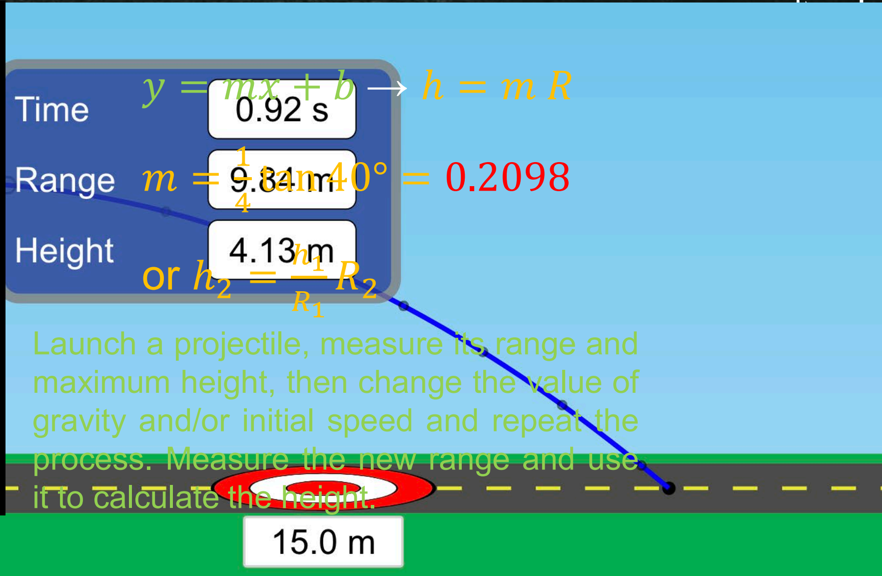
PhET free interactive math and science simulations

Determine the maximum height from the range and launch angle of a projectile.

$$h = \frac{R}{4}\tan\theta \qquad h = \frac{v_0^2 \cdot \sin^2(\theta)}{2g} \qquad R = \frac{v_0^2 \cdot \sin(2\theta)}{g}$$

Gravity	I
9.81	
7.00	
7.00	
5.00	





$y = mx + b \rightarrow h = mR$

$m = \frac{1}{4}\tan 40° = 0.2098$

or $h_2 = \frac{h_1}{R_1} R_2$

Launch a projectile, measure its range and maximum height, then change the value of gravity and/or initial speed and repeat the process. Measure the new range and use it to calculate the height.

Explore the Relationship Between Launch Angle Pairs and Initial Speed-to-Range Ratios:

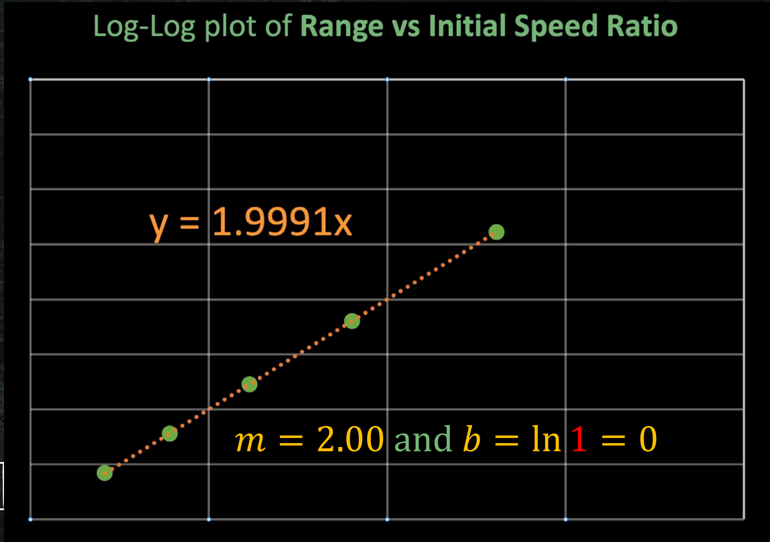
Use different complementary launch angle pairs (angles that sum to 90°) to observe how the ratio of initial speeds relates to the ratio of their corresponding ranges. Test this by calculating and comparing the ranges and initial speeds required for different angles.

$$\frac{v_1}{v_2} = \sqrt{\frac{R_1}{R_2}} \text{ or } \frac{R_1}{R_2} = \left(\frac{v_1}{v_2}\right)^2 \qquad R_i = \frac{v_i^2 \cdot \sin(2\theta_i)}{g} \text{ and } \theta_1 + \theta_2 = 90°$$

$\theta_1 = 25°, \theta_2 = 65°$

v_1	v_2	R_1	R_2	v_1/v_2	R_1/R_2	$\ln(v_1/v_2)$	$\ln(R_1/R_2)$
5	10	3.83	15.32	0.50	0.25	-0.69	-1.39
5	15	3.83	34.47	0.33	0.11	-1.10	-2.20
5	20	3.83	61.28	0.25	0.06	-1.39	-2.77
5	25	3.83	95.76	0.20	0.04	-1.61	-3.22
3	18	1.38	49.45	0.17	0.03	-1.79	-3.58

$$y = a x^n \rightarrow y = \ln y = n \ln x + \ln a = m x + b$$



A projectile is launched at an angle θ_1 with an initial speed v_1, while a second projectile is launched at θ_2, where $\theta_1 + \theta_2 = \frac{\pi}{2}$. The second projectile's initial speed is four times that of the first. What is the ratio of their ranges $\frac{R_1}{R_2}$?

Note that we have several options for modifying the above question due to the following relationships. One of the equalities holds when the angles are not complementary, and one of the equalities holds when gravity is not the same.

$$\frac{R_1}{R_2} = \left(\frac{v_1}{v_2}\right)^2, \frac{h_1}{h_2} = \left(\frac{t_1}{t_2}\right)^2, \frac{h_1}{h_2} = \frac{R_1}{R_2}\tan^2\theta_1$$

Examples of complementary angles are (37°, 53°) and (43°, 47°).
Find the remaining pairs $\theta_1 < \theta_2$ that follow this pattern.

References:
J. Fiala, “Projectile Motion: More Geometry, More Physics,” *Phys. Teach. 62, 210–213 (2024).*
https://doi.org/10.1119/5.0094435

Questions?

Start with $(\theta_1, \theta_2) = (7°, 83°)$ add to the first and subtract from the second $4, 6, 2, 4, 6, 2, 6, 6$.

$(\theta_1, \theta_2) = (7° + 4°, 83° - 4°) = (11°, 79°), \ldots,$ to get all 9 pairs!

4,6,2,4 | 6,2,6,6 is there a pattern or not?