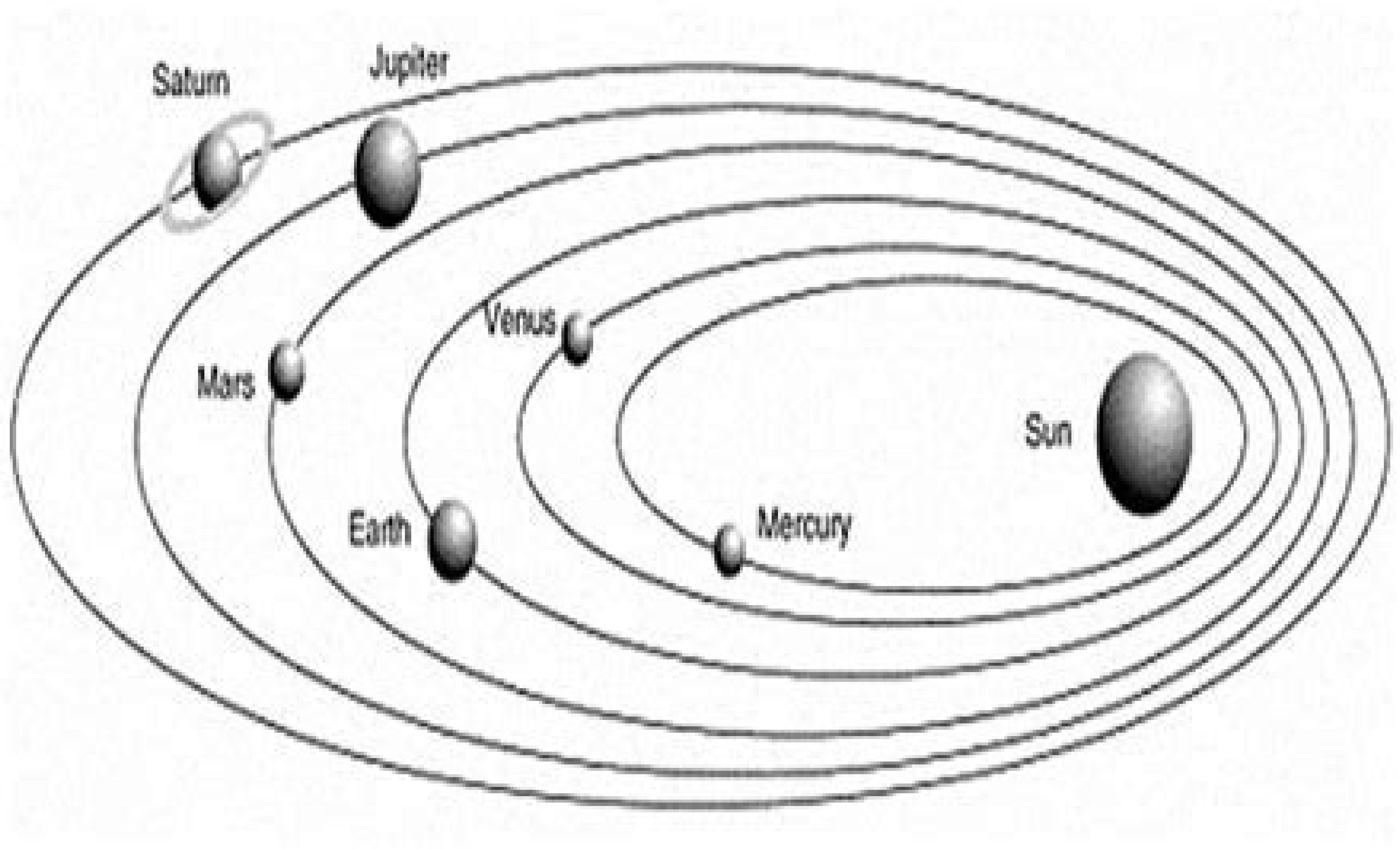
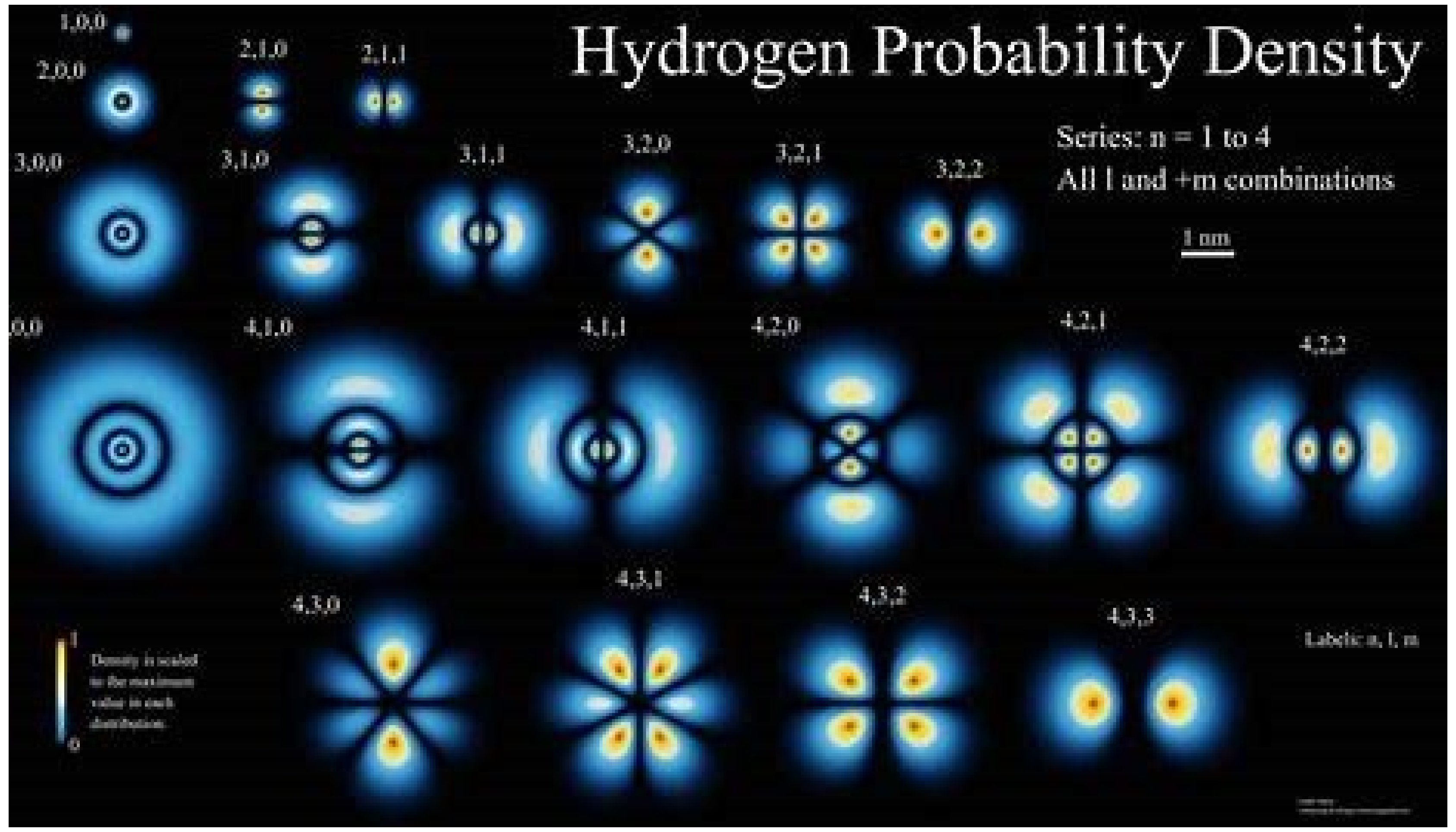
A simple approach to determining planetary orbits

Kepler's Elliptical Orbits



Quantum solutions



James Freericks, Department of Physics
Georgetown University
Work funded by the AFOSR and Georgetown

The conventional way to determine Kepler orbits is complicated. For example, we sketch how Goldstein does this.

Goldstein derivation

Hamilton equation of motion: $\dot{r} = \frac{p_r}{m}$ and $\dot{p_r} = F_{eff} = \frac{L^2}{mr^3} - \frac{k}{r^2}$, $\dot{\theta} = \frac{L}{mr^2}$, and $\dot{L} = 0$

So, we have $\ddot{r} = \frac{L^2}{m^2r^3} - \frac{k}{mr^2}$. Multiply by $\dot{r}$ and integrate to yield: $\frac{1}{2}\dot{r}^2 = E - \frac{L^2}{2m^2r^2} + \frac{k}{mr}$, with E the energy (arising as an integration constant).

We have $\frac{mr^2}{L}\, d\theta = dt$, so $\dot{r} = \sqrt{2E - \frac{L^2}{m^2r^2} + \frac{2k}{mr}} = \frac{dr}{d\theta}\frac{L}{mr^2}$, rearranging, we have

$$d\theta = dr \frac{L}{mr^2} \frac{1}{\sqrt{2E - \frac{L^2}{m^2r^2} + \frac{2k}{mr}}}$$

Goldstein derivation (part II)

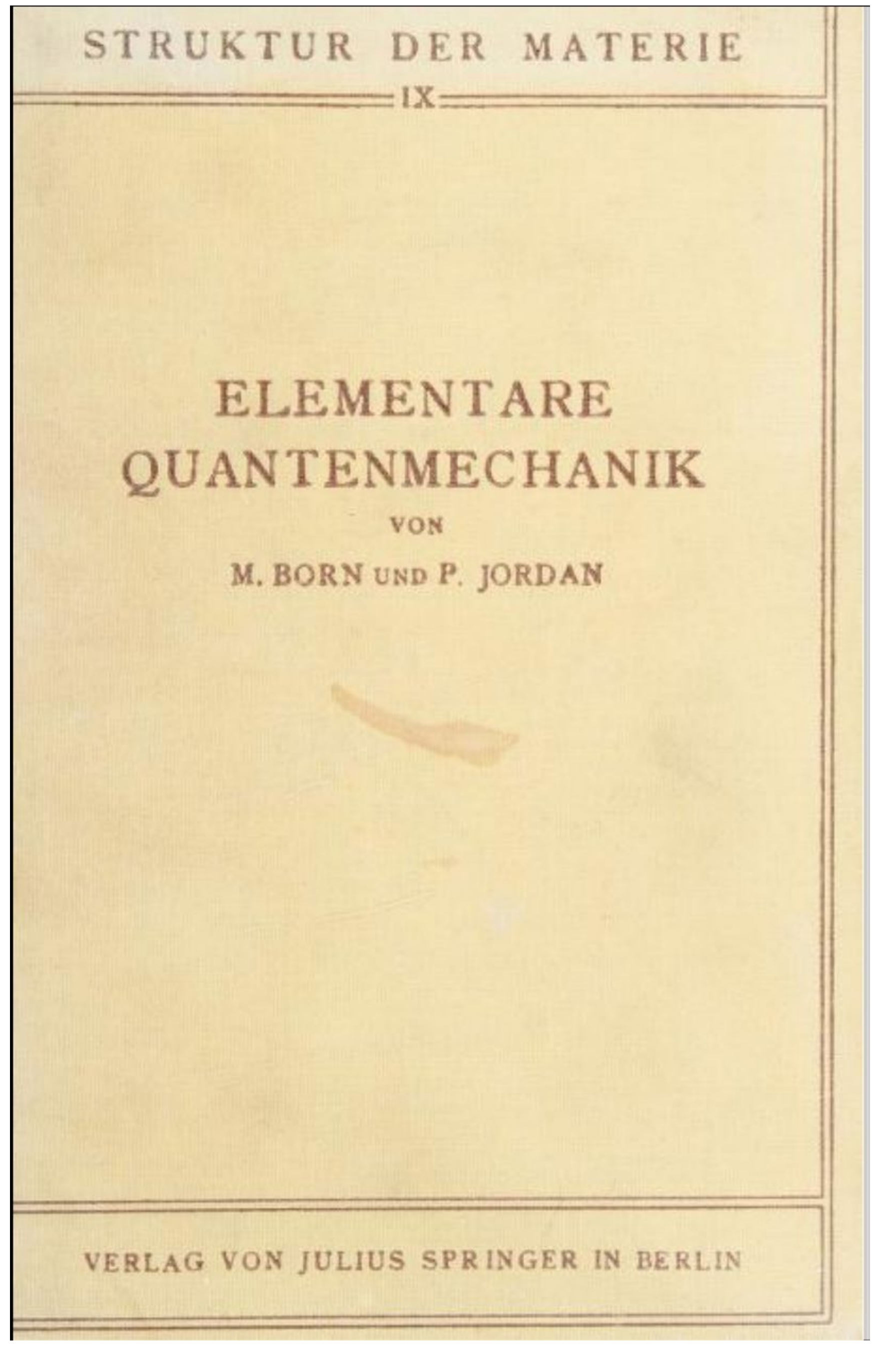
$$\int_{\theta_0} d\theta = \int_{r_0} dr \frac{L}{mr^2} \frac{1}{\sqrt{2E - \frac{L^2}{m^2 r^2} + \frac{2k}{mr}}}$$

$$\theta - \theta_0 = -\arccos \frac{\frac{L^2}{mkr} - 1}{\sqrt{1 + \frac{2EL^2}{mk^2}}}$$

So that

$$\frac{1}{r} = \frac{mk}{L^2}\left(1 + \sqrt{1 + \frac{2EL^2}{mk^2}} \cos(\theta - \theta_0)\right)$$

The idea for a simplification comes from an unlikely source: Born and Jordan's Elementare Quantenmechanik (1930)



Die kanonischen Bewegungsgleichungen

(3) $$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{\mu}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -a q$$

oder, mit der Abkürzung (2):

(3a) $$\ddot{q} = -(2\pi\nu_0)^2 q,$$

lauten wie in der klassischen Theorie. Definiert man „komplexe Amplituden"

(4) $$\begin{cases} b = C(p - 2\pi i \nu_0 \mu q), \\ b^\dagger = C(p + 2\pi i \nu_0 \mu q) \end{cases}$$

mit einer vorläufig beliebigen Konstanten C, so gehen die Gleichungen (3) über in

(5) $$\dot{b} = -2\pi i \nu_0 b, \qquad \dot{b}^\dagger = 2\pi i \nu_0 b^\dagger.$$

New derivation

Hamilton equation of motion: $\dot{r} = \frac{p_r}{m}$ and $\dot{p_r} = F_{eff} = \frac{L^2}{mr^3} - \frac{k}{r^2}$, $\dot{\theta} = \frac{L}{mr^2}$, and $\dot{L} = 0$

Define $A = p_r + \frac{\alpha}{r} + \beta$, then choose α and β so that the radial equations of motion are decoupled:

$$\dot{A} = \dot{p}_r - \frac{\alpha}{r^2}\,\dot{r} = \frac{L^2}{mr^3} - \frac{k}{r^2} - \frac{\alpha}{mr^2}\,p_r = -\frac{\alpha}{mr^2}\left(p_r - \frac{L^2}{\alpha r} + \frac{mk}{\alpha}\right)$$

So that $\alpha = \pm iL$, $\beta = \mp i\frac{mk}{L}$, and $\dot{A} = \mp i\frac{L}{mr^2}\,A = \mp i\,\dot{\theta}\,A$. Then we have

$$A(\theta) = A_0 e^{i\theta}, \qquad A^*(\theta) = A_0^* e^{-i\theta}, \qquad \text{and } A(\theta) = p_r + \frac{iL}{r} - \frac{imk}{L}$$

New derivation (part II)

$$A(\theta) = A_0 e^{i\theta}, \qquad A^*(\theta) = A_0^* e^{-i\theta}, \qquad \text{and } A(\theta) = p_r + \frac{iL}{r} - \frac{imk}{L}$$

If we set the constants at the point of closest approach r_0 we have $A_0 = i\left(\frac{L}{r_0} - \frac{mk}{L}\right)$

We remove p_r by taking the difference:

$$\frac{iL}{r} - \frac{imk}{L} = \frac{1}{2}\big(A(\theta) - A^*(\theta)\big) = i\left(\frac{L}{r_0} - \frac{mk}{L}\right)\cos(\theta - \theta_0)$$

Or

$$\frac{1}{r} = \frac{mk}{L^2} + \left(\frac{1}{r_0} - \frac{mk}{L^2}\right)\cos(\theta - \theta_0)$$

Using this approach of decoupling the differential equations is a much simpler derivation, requiring no complex integrations

Energy

We have $\frac{p_r^2(t)}{2m}+\frac{L^2}{2mr^2}-\frac{k}{r}+\frac{mk^2}{2L^2}=\frac{1}{2m}A^*(t)A(t)=\frac{1}{2m}|A_0|^2=\text{constant}$

So, energy is conserved, automatically in this approach

Relation to quantum mechanics

We found $A = p_r + \frac{iL}{r} - \frac{imk}{L}$ and $A^* = p_r - \frac{iL}{r} + \frac{imk}{L}$

If we make them operators, with $[\hat{r}, \hat{p}_r] = i\hbar$, then

$$\frac{1}{2m}\hat{A}^\dagger\hat{A} = \frac{1}{2m}\left(\hat{p}_r - \frac{iL}{\hat{r}} + \frac{imk}{L}\right)\left(\hat{p}_r + \frac{iL}{\hat{r}} - \frac{imk}{L}\right) = \frac{\hat{p}_r{}^2}{2m} + \frac{iL}{2m}\left[\hat{p}_r, \frac{1}{\hat{r}}\right] + \frac{L^2}{2m\hat{r}^2} - \frac{k}{\hat{r}} + \frac{mk^2}{2L^2}$$

$$= \frac{\hat{p}_r{}^2}{2m} + \frac{L^2 + \hbar L}{2m\hat{r}^2} - \frac{k}{\hat{r}} + \frac{mk^2}{2L^2} = \hat{H} - E_l$$

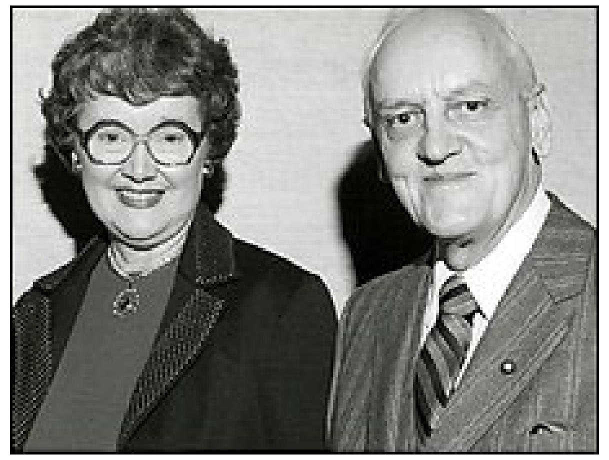
This is the solution of the hydrogen atom for a fixed angular momentum state, if we let $L = \hbar l$.

So, the simplified treatment of the classical orbit also provides a neat connection to the quantum solution.

Pretty cool.

Thank you Born and Jordan!

Thanks to



Leanne Doughty   Jason Tran