Spring 2024 Meeting

Saturday 19 October 2024

Coefficient of Performance (COP) of a Stirling Refrigerator

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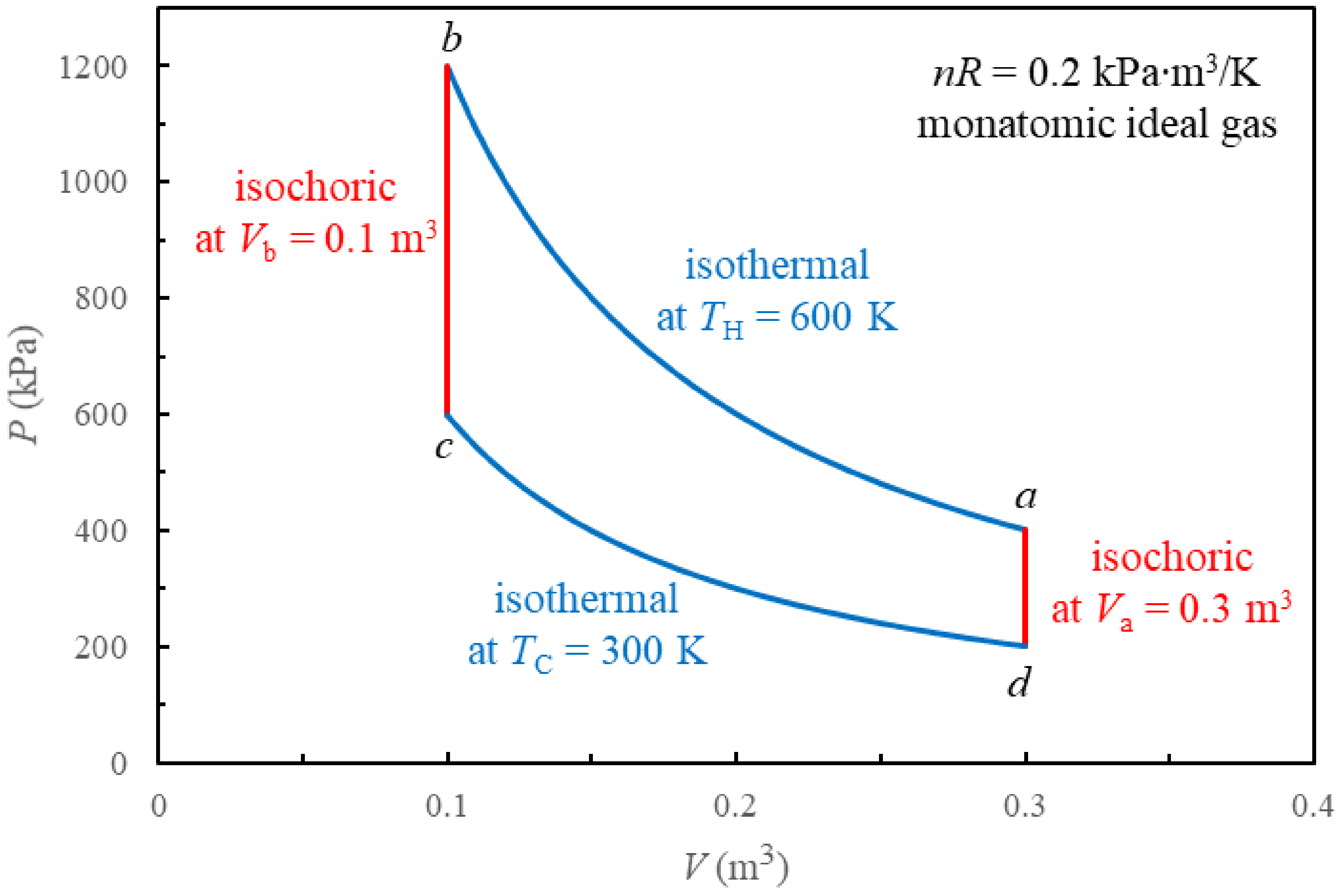
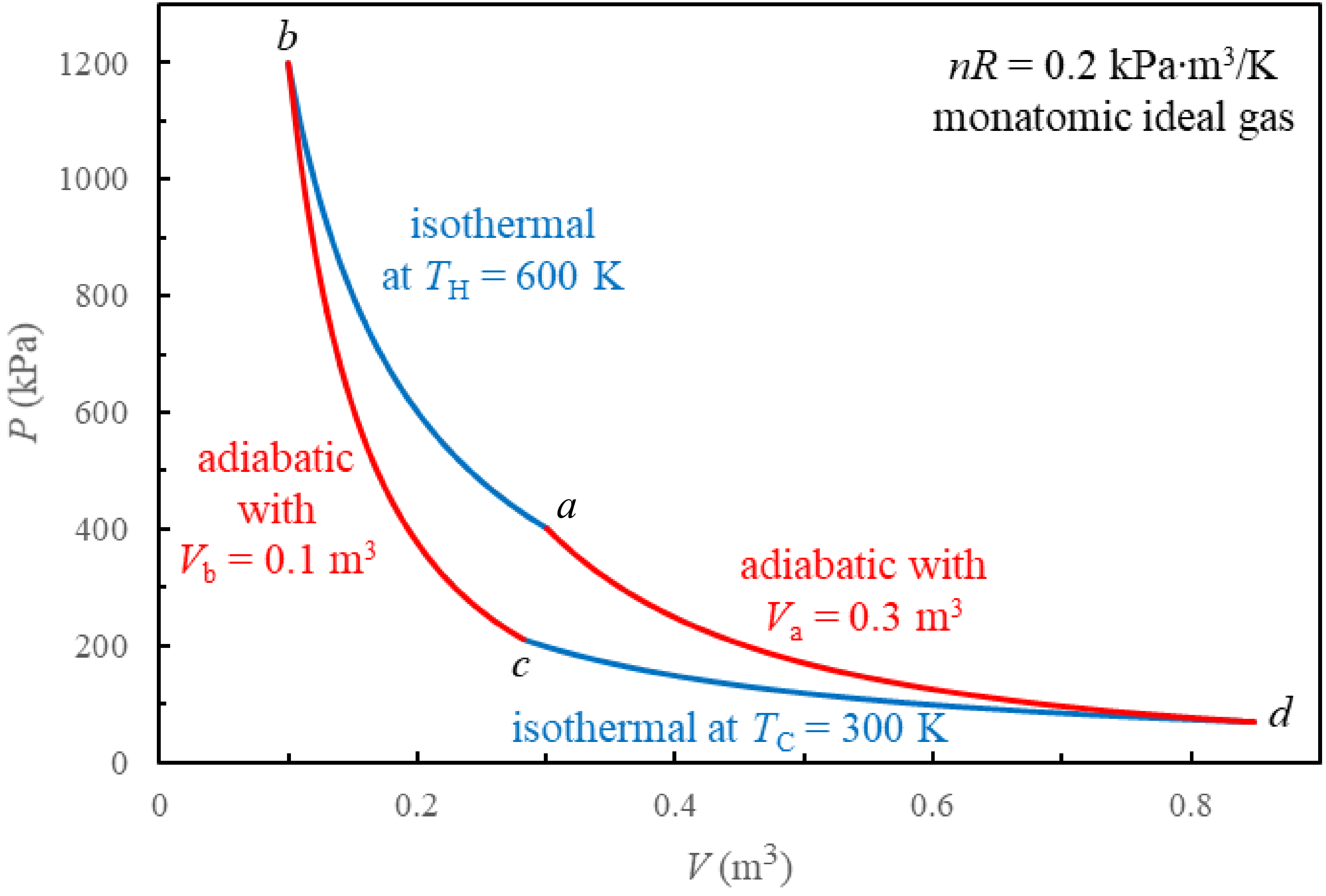
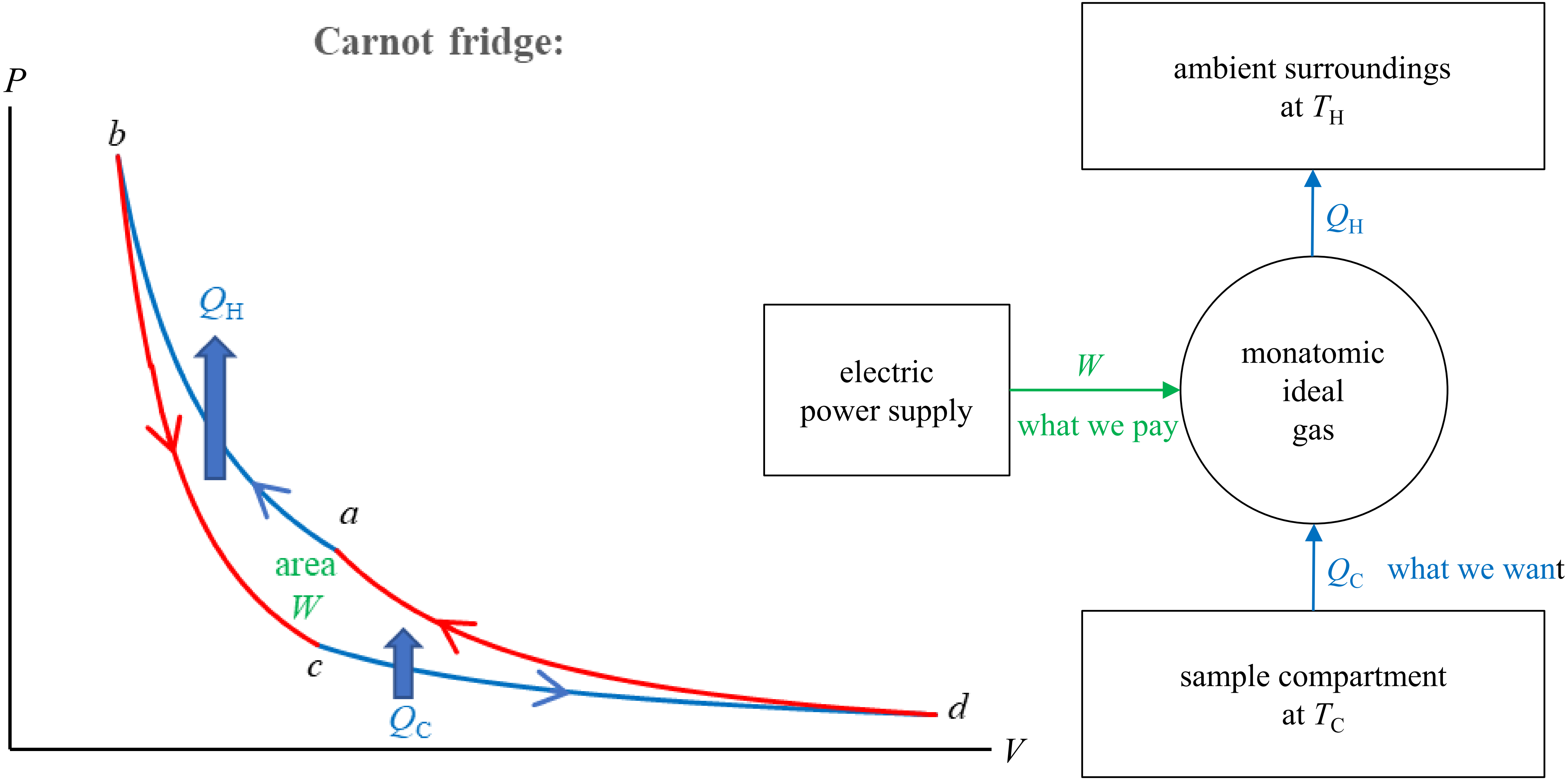
Stirling Cycle



Carnot Cycle



Carnot fridge:



ambient surroundings
at T_H

Q_H

electric
power supply

W

what we pay

monatomic
ideal
gas

Q_C what we want

sample compartment
at T_C

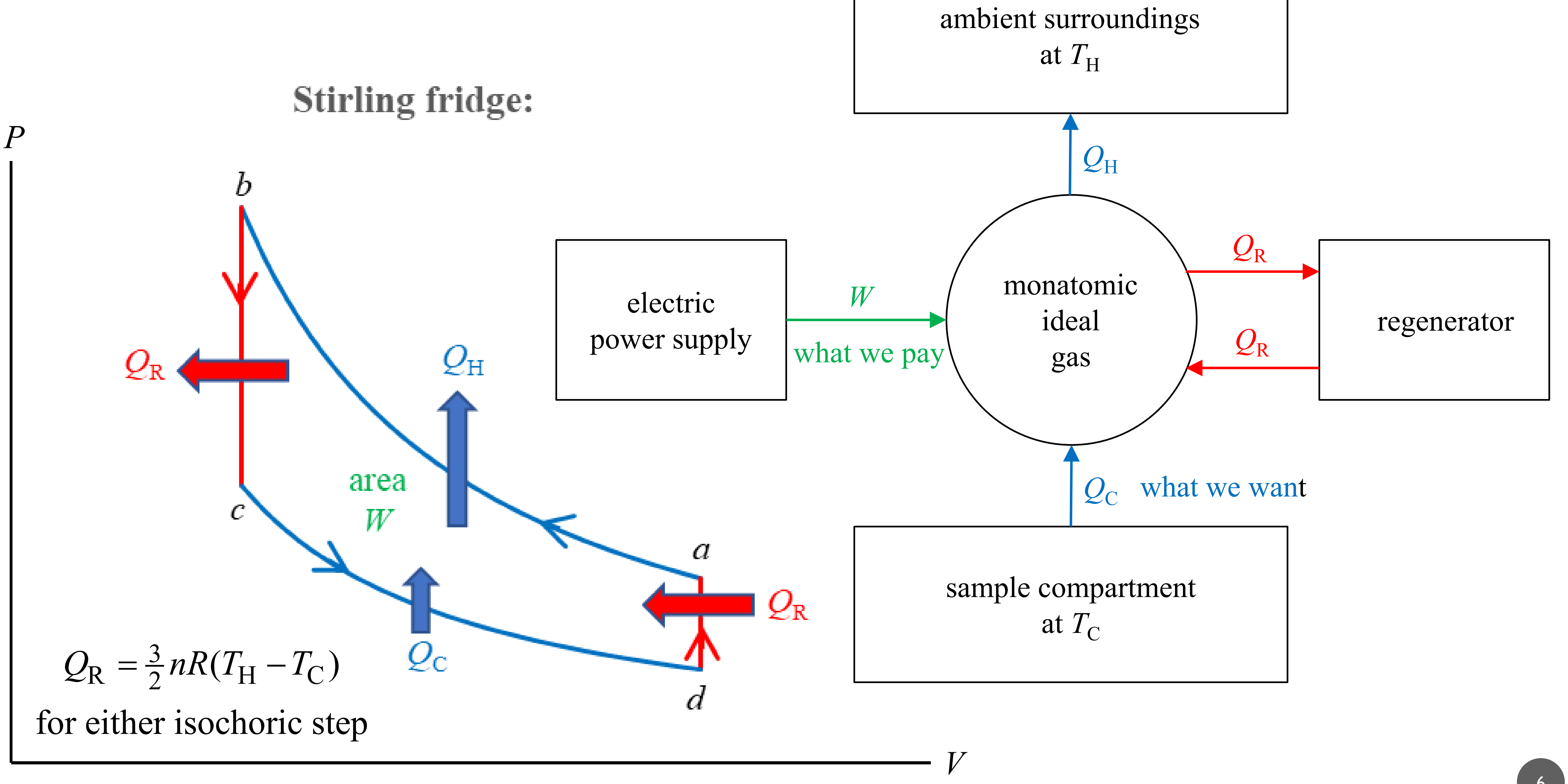
good refrigerator performance: lots of what we want relative to what we pay

$$\therefore\ \mathrm{COP} \equiv \frac{Q_\mathrm{C}}{W}$$

energy balance per cycle: $\sum \text{inputs to gas} = \sum \text{outputs from gas} \quad \Rightarrow \quad W + Q_\mathrm{C} = Q_\mathrm{H}$

$$\therefore\ \mathrm{COP}_\mathrm{Carnot} = \frac{Q_\mathrm{C}}{Q_\mathrm{H} - Q_\mathrm{C}} = \frac{T_\mathrm{C}}{T_\mathrm{H} - T_\mathrm{C}}$$

scheme #1—reversible with regenerator



Stirling fridge:

$Q_R = \frac{3}{2} nR(T_H - T_C)$

for either isochoric step

ambient surroundings at T_H

Q_H

electric power supply

W

what we pay

monatomic ideal gas

Q_R

Q_R

regenerator

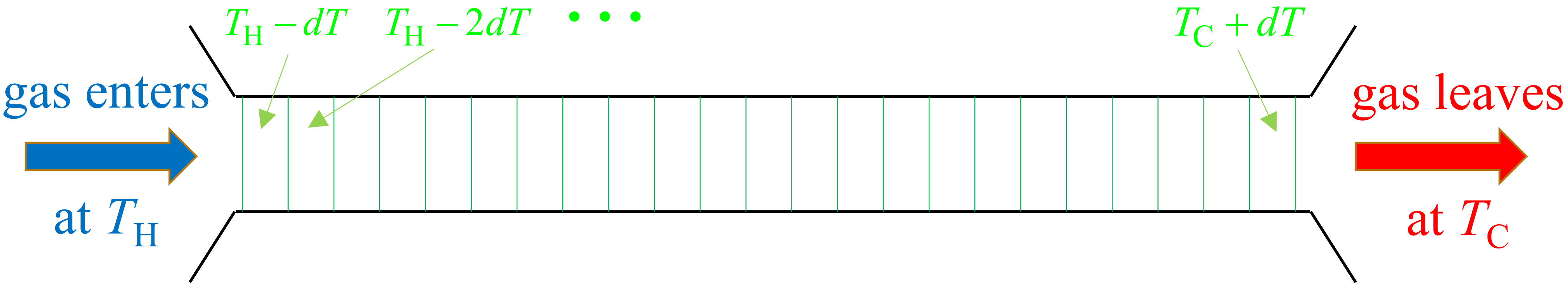
Q_C what we want

sample compartment at T_C

What is a regenerator?

It is a heat exchanger that stores and releases heat on demand.

Consider step bc when the ideal gas is cooled:



Reverse the flow for step da when the ideal gas is heated.

Thus the regenerator functions like a set of <u>intermediate temperature</u> reservoirs. (Ideally an infinite number of them.)

again what we want relative to what we pay:

$$\text{COP} \equiv \frac{Q_\text{C}}{W} = \frac{Q_\text{C}}{(Q_\text{H} + Q_\text{R}) - (Q_\text{C} + Q_\text{R})} = \frac{Q_\text{C}}{Q_\text{H} - Q_\text{C}}$$

but for an isothermal step:

$$Q_\text{in} = W_\text{out} \text{ since } \Delta U = 0 \qquad \Rightarrow Q_\text{C} = \int_{V_\text{c}}^{V_\text{d}} \frac{nRT_\text{C}}{V} dV = nRT_\text{C} \ln \frac{V_\text{d}}{V_\text{c}}$$

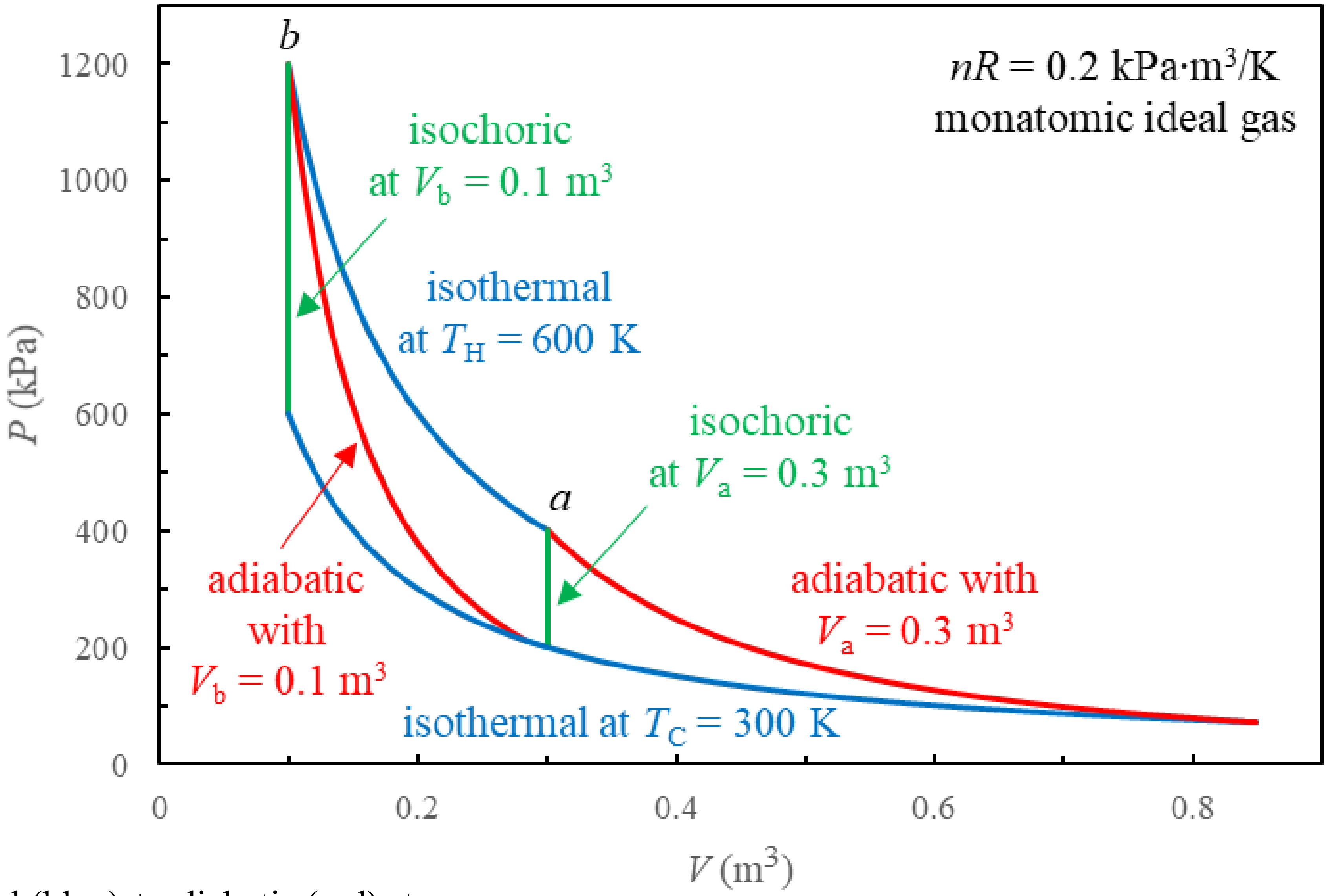
$$\text{likewise } Q_\text{H} = nRT_\text{H} \ln \frac{V_\text{a}}{V_\text{b}}$$

and $V_\text{a}/V_\text{b} = V_\text{d}/V_\text{c}$ for either isochoric (Stirling) or adiabatic (Carnot) connecting steps

since $TV^{2/3} = \text{constant}$

$$\therefore \ \text{COP}_\text{Stirling rev} = \frac{T_\text{C}}{T_\text{H} - T_\text{C}} = \text{COP}_\text{Carnot}$$

Carnot compared to Stirling

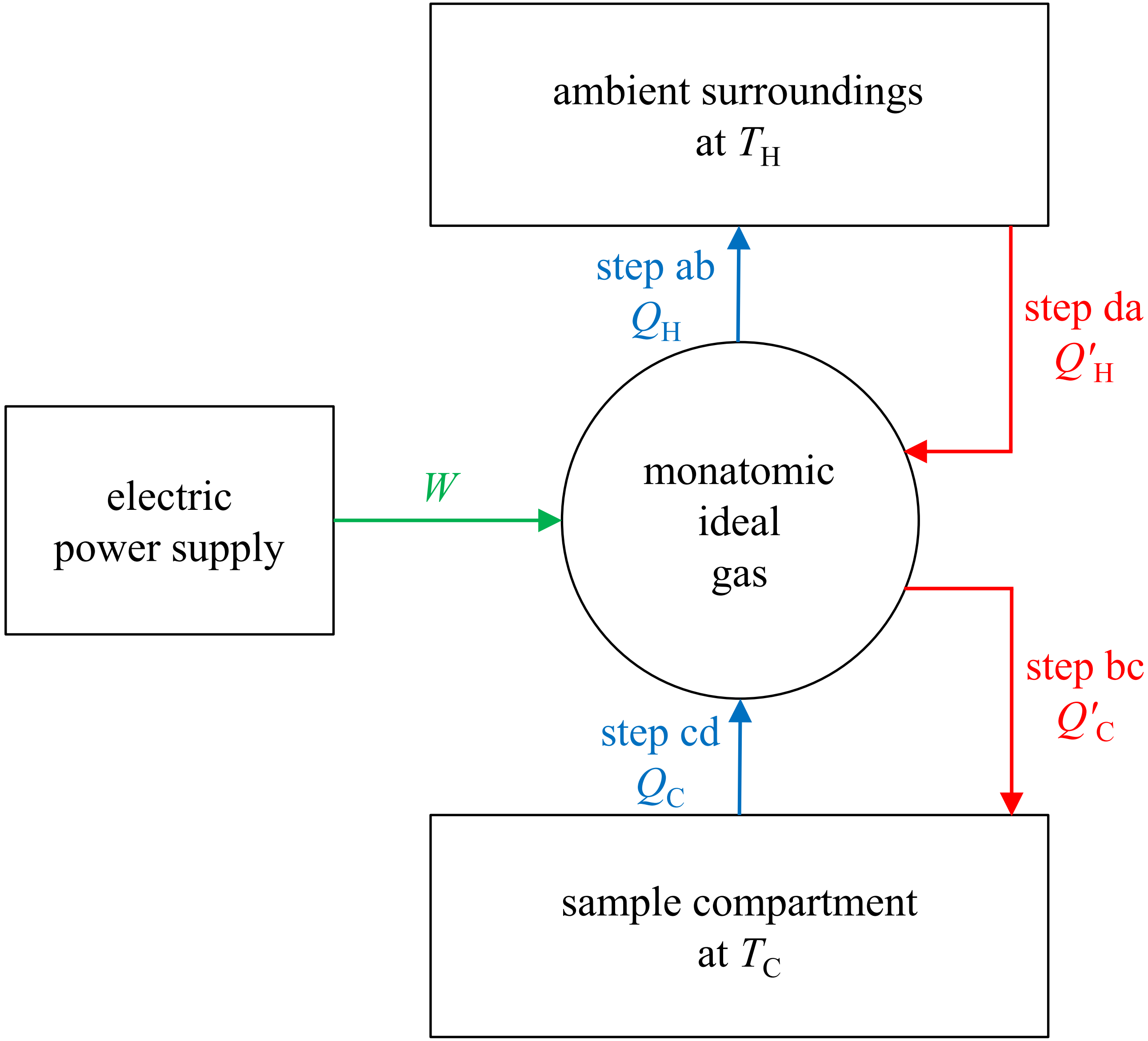


Carnot: isothermal (blue) + adiabatic (red) steps

Stirling: isothermal (blue) + isochoric (green) steps

same Q_H, Q_C, and W for the Carnot and Stirling cycles!

scheme #2—irreversible without regenerator



ambient surroundings at T_H

step ab Q_H

step da Q'_H

electric power supply

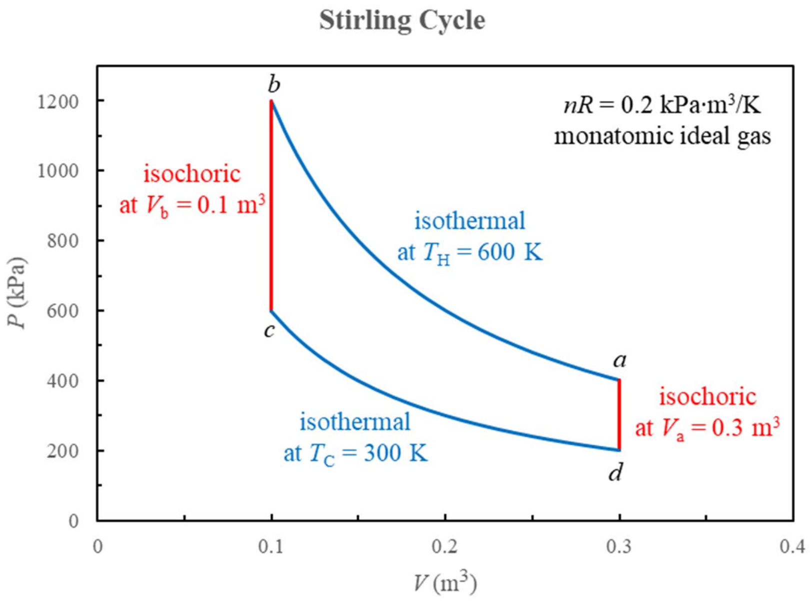
W

monatomic ideal gas

step bc Q'_C

step cd Q_C

sample compartment at T_C



$$\mathrm{COP} \equiv \frac{\text{net cooling of sample compartment}}{\text{total work done on ideal gas}} = \frac{Q_\mathrm{C} - Q'_\mathrm{C}}{(Q_\mathrm{H} - Q'_\mathrm{H}) - (Q_\mathrm{C} - Q'_\mathrm{C})}$$

isothermal steps $\begin{cases} Q_\mathrm{H} = nRT_\mathrm{H} \ln\left(V_\mathrm{d} / V_\mathrm{c}\right) \\ Q_\mathrm{C} = nRT_\mathrm{C} \ln\left(V_\mathrm{d} / V_\mathrm{c}\right) \end{cases}$ isochoric steps $\begin{cases} Q'_\mathrm{H} = \frac{3}{2} nR(T_\mathrm{H} - T_\mathrm{C}) \\ Q'_\mathrm{C} = \frac{3}{2} nR(T_\mathrm{H} - T_\mathrm{C}) \end{cases}$

$$\therefore \mathrm{COP} = \frac{nRT_\mathrm{C} \ln\left(V_\mathrm{d} / V_\mathrm{c}\right) - \frac{3}{2} nR(T_\mathrm{H} - T_\mathrm{C})}{nR\left(T_\mathrm{H} - T_\mathrm{C}\right) \ln\left(V_\mathrm{d} / V_\mathrm{c}\right)} = \frac{T_\mathrm{C}}{T_\mathrm{H} - T_\mathrm{C}} - \frac{1.5}{\ln\left(V_\mathrm{d} / V_\mathrm{c}\right)}$$

the COP is <u>reduced</u> from the Carnot value
because of the irreversible isochoric steps

Comments and questions are welcome!

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where you can find further results in
EJP **38**, 055101 (2017) and **41**, 058002 (2020)