From Discrete Pattern to Continuous Motion

Galileo's Law of Odd Numbers

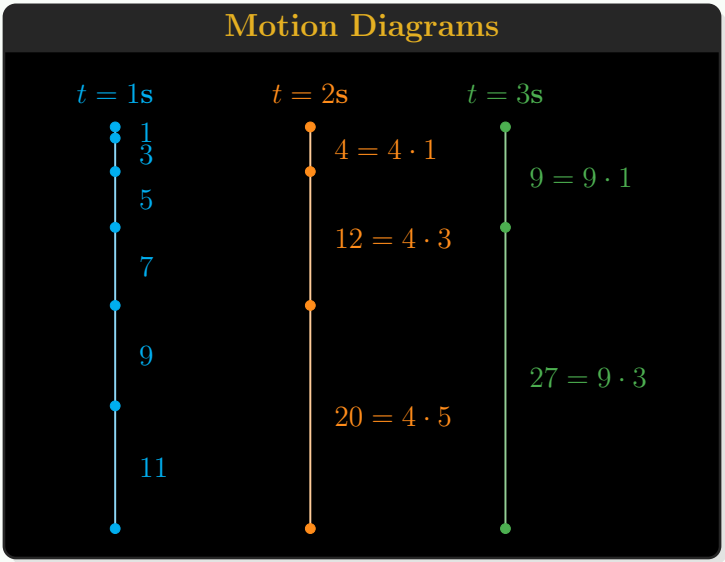
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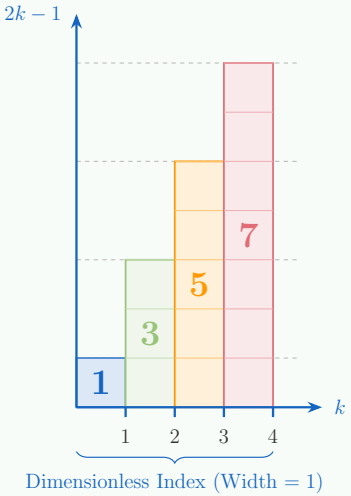
Fall Meeting of the Chesapeake Section of the AAPT
College of William & Mary, Williamsburg VA
Saturday 10 October 2026

Observation:

- Galileo's Scaling: Video



Abstracting the Pattern: Pure Geometry



First examine the **dimensionless** sequence $1, 3, 5, 7, \ldots$: each column has width 1 and height $2k-1$. Each little square is one unit of area.

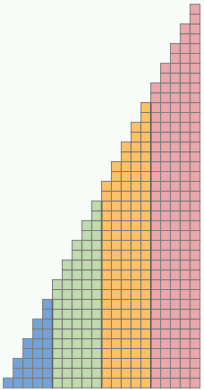
▸ geometry of odd numbers

At this stage, the pattern is **dimensionless**.

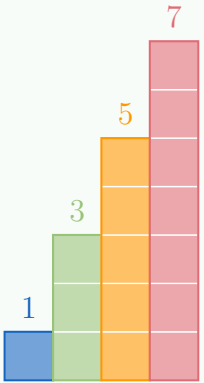
Later: width $\to$ time, height $\to$ average speed, area $\to$ displacement.

Example: Bundling Five Columns

Group five consecutive columns of the same color. **Each new rectangle has exactly the same area** as its original group.



Five columns per color



Each cell: 5×5

The Pattern Survives

$$25(1, 3, 5, 7, \ldots)$$

$$\sum_{k=m(j-1)+1}^{mj} (2k-1) = m^2(2j-1)$$

Numerical check ($m = 5$):

$1 + 3 + 5 + 7 + 9 = 25 = 5^2(1)$

$11 + 13 + 15 + 17 + 19 = 75 = 5^2(3)$

$21 + 23 + 25 + 27 + 29 = 125 = 5^2(5)$

$31 + 33 + 35 + 37 + 39 = 175 = 5^2(7)$

Physical Mapping: From Geometry to Motion

So far, we have only counted unit squares. Now assign **physical units** using two physical scales: time t for one interval and displacement d during the first interval.

One Geometric Square Becomes a Physical Displacement

$$\underbrace{1}_{\text{width}} \times \underbrace{1}_{\text{height}} \longrightarrow \underbrace{\mathbf{t}}_{\text{s}} \times \underbrace{\mathbf{v}}_{\text{m/s}} = \underbrace{\mathbf{d}}_{\text{m}}$$

Define $v = d/t$, the **average speed in the first interval**. During the k-th equal interval:

$$\Delta t_k = \mathbf{t} \quad \text{(same observation interval)},$$

$$\Delta x_k = (2k-1)\mathbf{d} \quad \text{(observed odd-number pattern)},$$

$$\bar{v}_k = \frac{\Delta x_k}{\Delta t_k} = (2k-1)\frac{\mathbf{d}}{\mathbf{t}} = (2k-1)\mathbf{v}.$$

Rectangle width $\times$ average-speed height $=$ displacement: this is an extended motion diagram, not yet a continuous velocity graph.

Scaling Physical Units: What the Geometry Forces

Fine Observation Scale

One clock tick is t; the first displacement is d.

- Successive displacements: $d, 3d, 5d, \ldots$

Coarse Observation Scale

Combine m fine intervals: $T = mt$.

- The first coarse displacement is D.

The first coarse interval consists of m fine intervals. Their displacements **must add to the same measured displacement**:

$$D = \sum_{k=1}^{m} (2k-1)d = m^2 d$$

We choose how many intervals to group; the odd-number geometry determines D.

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Interactive Scaling: Observe More Frequently

Simulation: Scaling & Average Speeds

Refining the Observation Interval

We **choose** an observation interval m times shorter. To describe the **same motion** with the same odd-number pattern, the remaining scales are determined:

$$t_{\text{new}} = \frac{t}{m} \qquad \text{(chosen clock)},$$

$$d_{\text{new}} = \frac{d}{m^2} \qquad \text{(geometry)},$$

$$v_{\text{new}} = \frac{d_{\text{new}}}{t_{\text{new}}} = \frac{v}{m} \qquad \text{(average-speed scale)}.$$

The smaller v_{new} is the average speed in the *first shorter interval*—not a decrease in the object's speed at a fixed later instant.

Summary: What Did We Assume, Choose, and Derive?

Scaling Rules

Starting Point / Step	Result	Origin
Observation	$1, 3, 5, 7, \ldots$	*Measured displacements*
Geometry	$\sum_{k=1}^{m}(2k-1) = m^2$	*Counting unit squares*
Clock refinement	$t_{\text{new}} = t/m$	*Our choice*
Displacement scale	$d_{\text{new}} = d/m^2$	*Forced by geometry*
Average-speed scale	$v_{\text{new}} = v/m$	*Distance/time*
Continuous limit	$x(s) - x(0) = Cs^2$	*Refinement + continuity*

Geometry determines scaling; **units** connect it to motion; **refinement** approaches continuity.

Thank You

Questions?

Reconstructing Continuous Motion

We have not assumed an equation of motion. Suppose the odd-number pattern holds consistently when we refine the time intervals, with position continuous between observations.

From Odd Numbers to Position

After n intervals of duration t:

$$x(nt) - x(0) = d\sum_{k=1}^{n}(2k-1) = n^2 d.$$

Refinement keeps d/t^2 unchanged. Define $C = d/t^2$:

$$\boxed{x(s) - x(0) = Cs^2}.$$

Here s is elapsed physical time, to distinguish it from the interval size t.

From Averages to an Instant

At a fixed elapsed time $s = nt$, the average over the preceding interval is

$$\bar{v}_n = (2n-1)\frac{d}{t} = C(2s - t).$$

As $t \to 0$ while s stays fixed,

$$\boxed{v(s) = 2Cs}.$$

The index $n = s/t$ grows as we refine the clock.

One finite data set is not enough; the continuous reconstruction also uses scale-consistency and continuity.