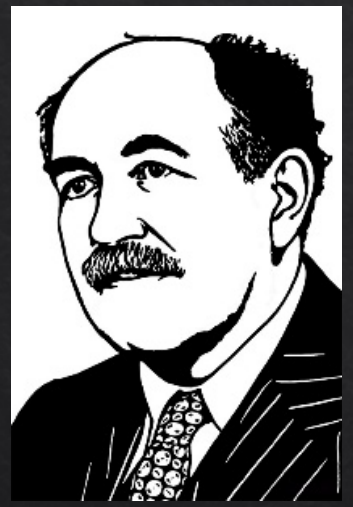
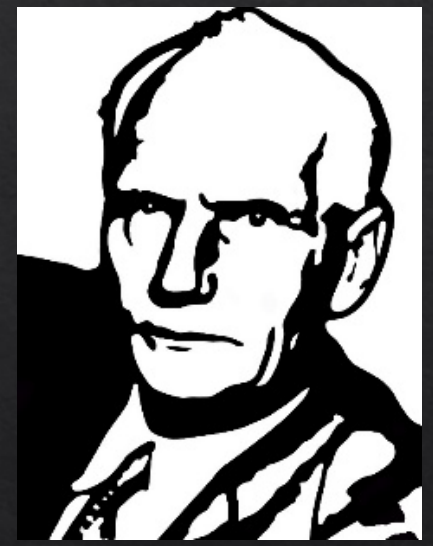
An experimental-driven introduction to developing the quantum formalism for spin in a Modern Physics class

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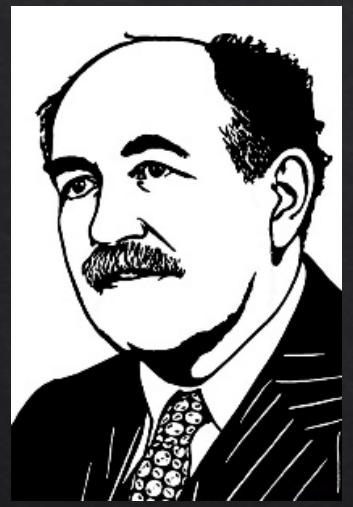
The spins-first paradigm (classical)

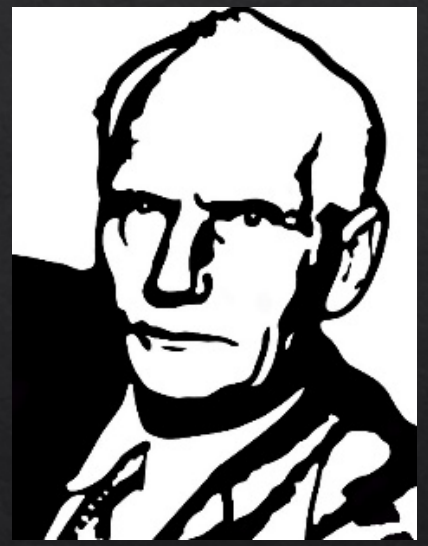




Classically, the spins are separated according to the projection of their effective magnetic field along the axis of increasing magnetic field

The spins-first paradigm (quantum)





Quantum mechanically, the spins are separated into one of two projections regardless of the orientation of the magnetic field

We call these states $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$

Now we can start to introduce the spin formalism

We create the operator that yields the measured projection of the spin

So

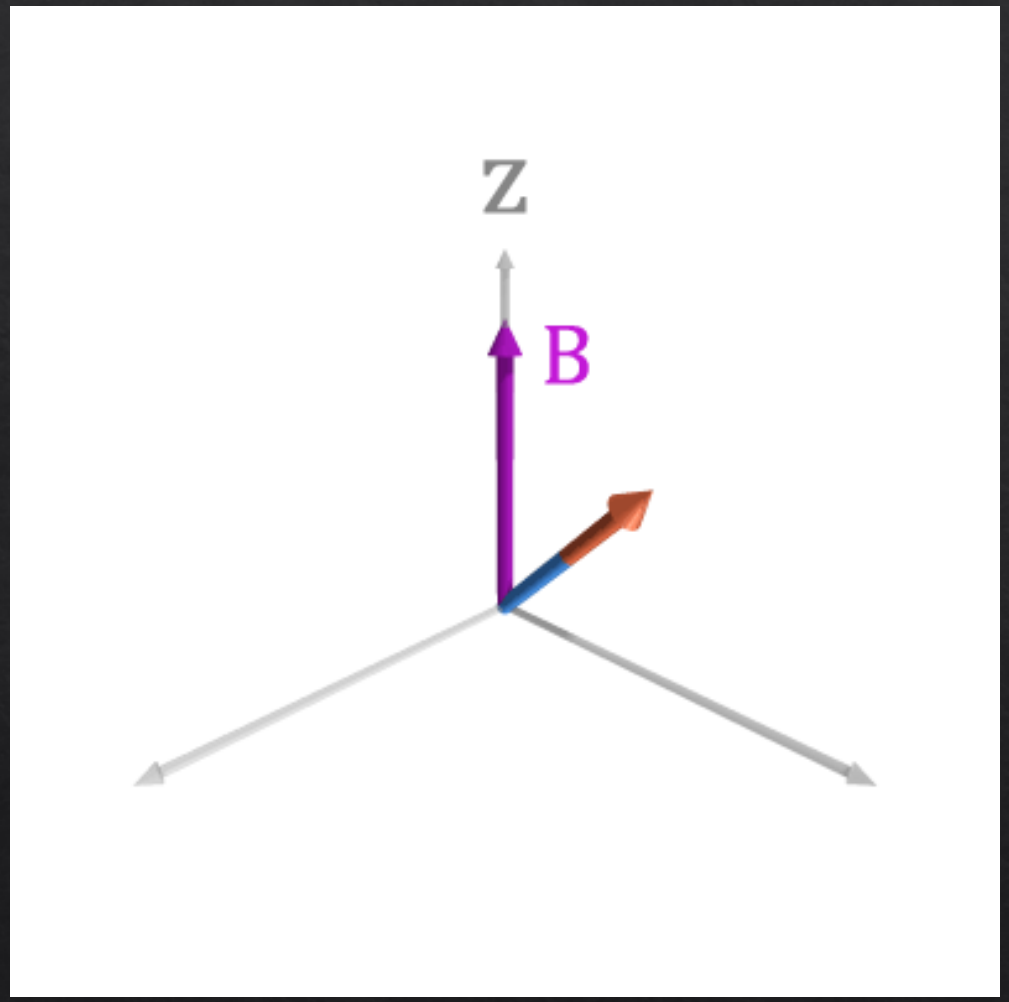
$$\hat{S}_z|\uparrow\rangle_z = \frac{\hbar}{2}|\uparrow\rangle_z$$

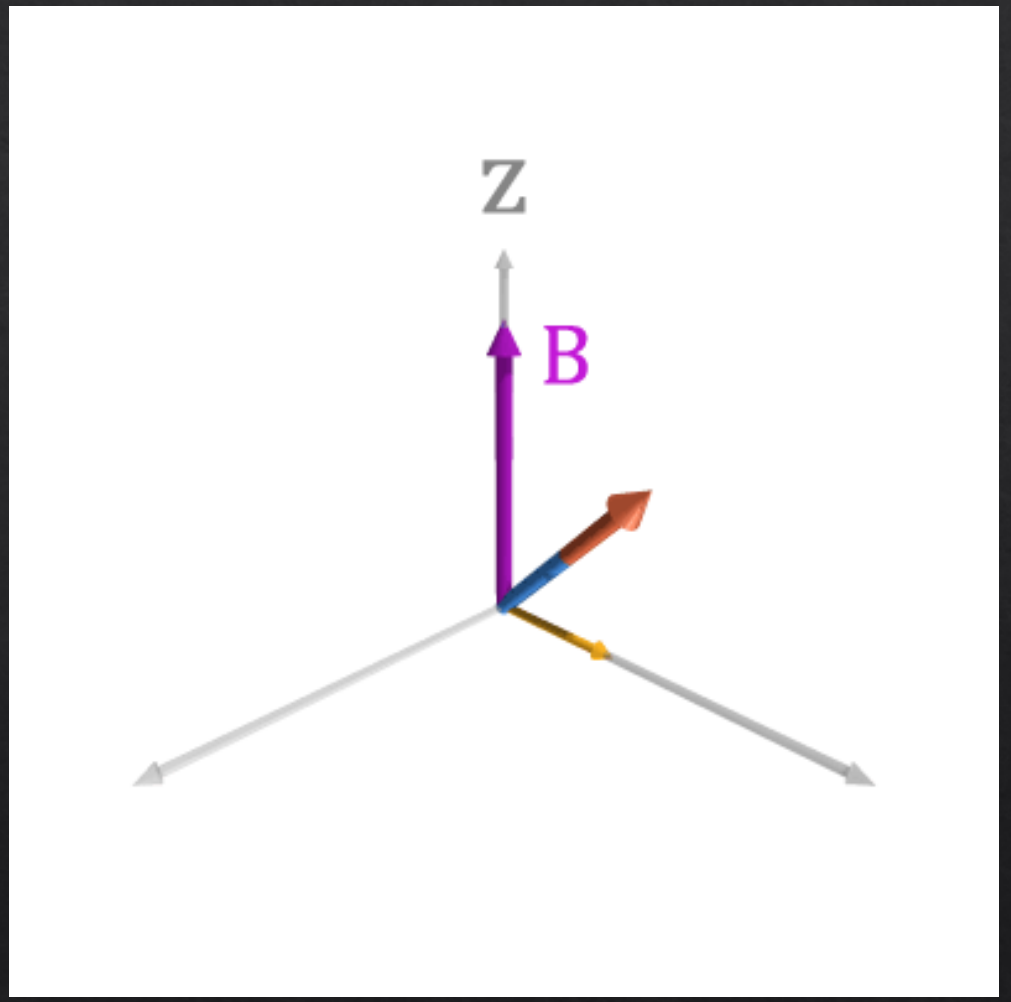
and

$$\hat{S}_z|\downarrow\rangle_z = -\frac{\hbar}{2}|\downarrow\rangle_z.$$

Note that these are eigenvalue-eigenvector relations, or, in other words, they correspond to states with definite projections of spin along the z-axis.

Next, we consider magnetic resonance





Magnetic resonance allows us to flip a spin

So we need an operator that converts $|\uparrow\rangle_z \rightarrow |\downarrow\rangle_z$ and *vice versa*.

The spin raising operator

So

$$\hat{S}_+|\downarrow\rangle_z = \hbar|\uparrow\rangle_z$$

and

$$\hat{S}_+|\uparrow\rangle_z = 0.$$

We use $\hbar$ in this spin-1/2 case, because it is equal to the difference of the two eigenvalues for the z-operator.
In the second equation, the raising operator annihilates the state, because the Stern-Gerlach experiment has only two states, so the up state cannot be raised.

The spin lowering operator

So

$$\hat{S}_-|\uparrow\rangle_z = \hbar|\downarrow\rangle_z$$

and

$$\hat{S}_-|\downarrow\rangle_z = 0.$$

We use $\hbar$ in this spin-1/2 case, because it is equal to the difference of the two eigenvalues for the z-operator.
In the second equation, the lowering operator annihilates the state, because the Stern-Gerlach experiment has only two states, so the down state cannot be lowered.

The SU(2) commutator algebra (I)

We find $\hat{S}_z\hat{S}_+|\uparrow\rangle_z = 0$ and $\hat{S}_+\hat{S}_z|\uparrow\rangle_z = 0$, so that $\left[\hat{S}_z, \hat{S}_+\right]|\uparrow\rangle_z = 0$ and $\hat{S}_z\hat{S}_+|\downarrow\rangle_z = \frac{\hbar^2}{2}|\uparrow\rangle_z$ and $\hat{S}_+\hat{S}_z|\downarrow\rangle_z = -\frac{\hbar^2}{2}|\uparrow\rangle_z$, so that $\left[\hat{S}_z, \hat{S}_+\right]|\downarrow\rangle_z = \hbar^2|\uparrow\rangle_z$.

Looking at the action of the commutator on both basis states, we see it can be written in an operator form as

$$\left[\hat{S}_z, \hat{S}_+\right] = \hbar\hat{S}_+$$

The SU(2) commutator algebra (II)

We find $\hat{S}_z\hat{S}_-|\uparrow\rangle_z = -\frac{\hbar^2}{2}|\downarrow\rangle_z$ and $\hat{S}_-\hat{S}_z|\uparrow\rangle_z = \frac{\hbar^2}{2}|\downarrow\rangle_z$, so that $\left[\hat{S}_z, \hat{S}_-\right]|\uparrow\rangle_z = -\hbar^2|\downarrow\rangle_z$

and $\hat{S}_z\hat{S}_-|\downarrow\rangle_z = 0$ and $\hat{S}_-\hat{S}_z|\downarrow\rangle_z = 0$, so that $\left[\hat{S}_z, \hat{S}_-\right]|\downarrow\rangle_z = 0$.

Looking at the action of the commutator on both basis states, we see it can be written in an operator form as

$$\left[\hat{S}_z, \hat{S}_-\right] = -\hbar\hat{S}_-$$

The SU(2) commutator algebra (III)

We find $\hat{S}_+\hat{S}_-|\uparrow\rangle_z = \hbar^2|\uparrow\rangle_z$ and $\hat{S}_-\hat{S}_+|\uparrow\rangle_z = 0$, so that $[\hat{S}_+, \hat{S}_-]|\uparrow\rangle_z = \hbar^2|\uparrow\rangle_z$
and $\hat{S}_+\hat{S}_-|\downarrow\rangle_z = 0$ and $\hat{S}_-\hat{S}_+|\downarrow\rangle_z = \hbar^2|\downarrow\rangle_z$, so that $[\hat{S}_+, \hat{S}_-]|\downarrow\rangle_z = -\hbar^2|\downarrow\rangle_z$.

Looking at the action of the commutator on both basis states, we see it can be written in an operator form as

$$[\hat{S}_+, \hat{S}_-] = 2\hbar\hat{S}_z$$

So we derive the SU(2) commutation relations from the definitions of the three operators!

Hence, one can develop the spin formalism by relying on experimental results.

It is forced on us by wanting to describe the Stern-Gerlach experiment and the magnetic resonance experiment.

The Cartesian operators

We define

$$\hat{S}_x = \frac{\hat{S}_+ + \hat{S}_-}{2}$$

and

$$\hat{S}_y = \frac{\hat{S}_+ - \hat{S}_-}{2i}$$

The Cartesian operators

Using the SU(2) commutation relations, one can now show that

$$[\hat{S}_i, \hat{S}_j] = i\hbar \sum_k \epsilon_{ijk}\, \hat{S}_k$$

which is the Cartesian form of the commutation relations.

This allows us to introduce the spin formalism by using the Stern-Gerlach experiment and the magnetic resonance experiment.

We do not need to just postulate the results.

From these spin operators, we can find their matrix representations, which yields the Pauli matrices, and eventually, the Pauli matrix identities, such as the generalized Euler identity.



We believe this is a more effective way to introduce formalism to the students in the spins-first paradigm.

This work is part of a larger project to revamp the way we teach quantum mechanics to sophomores.

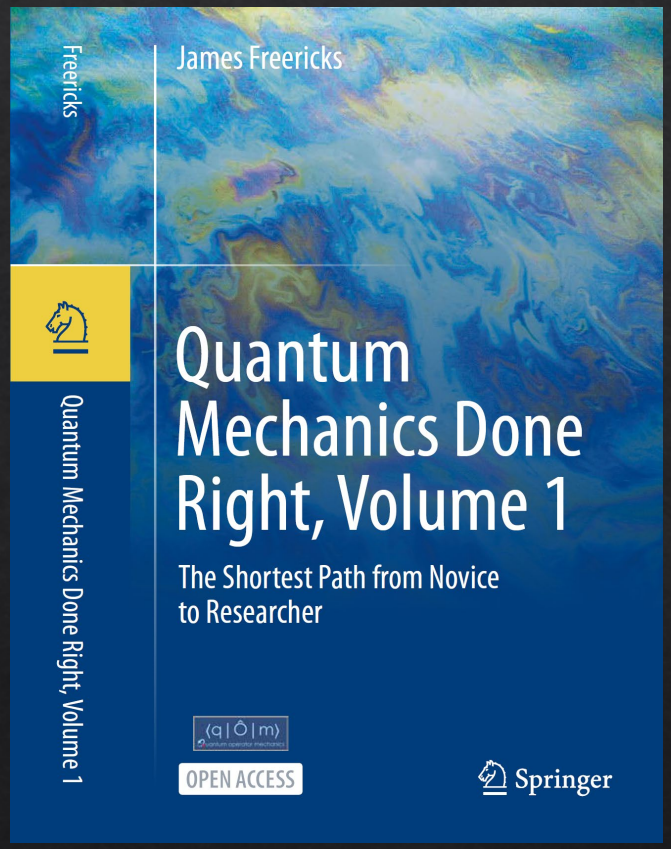


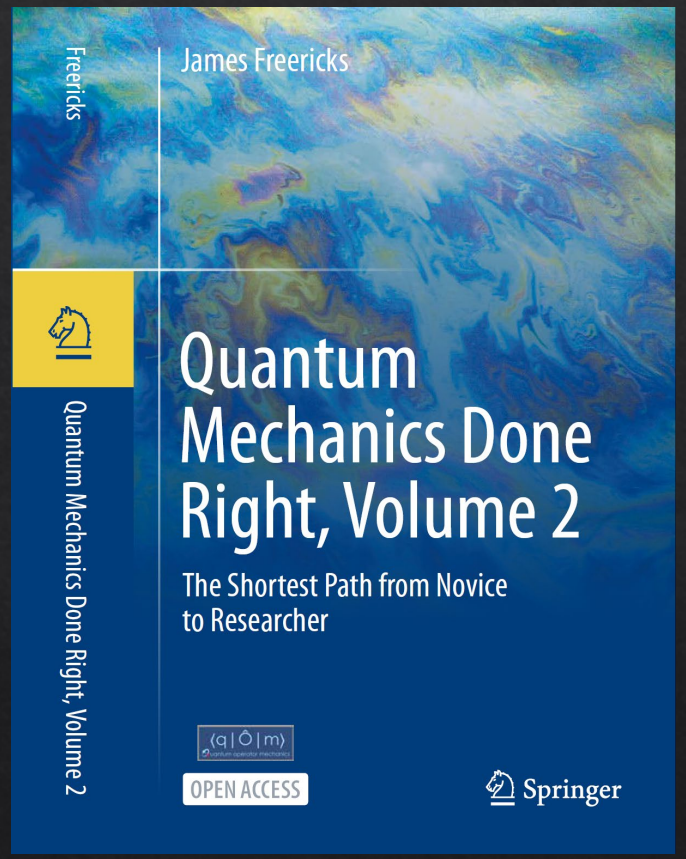
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AAPT Summer Meeting