Fall Meeting of the Chesapeake Section of the American Association of Physics Teachers

WHEN THE IRREVERSIBLE BECOMES REVERSIBLE

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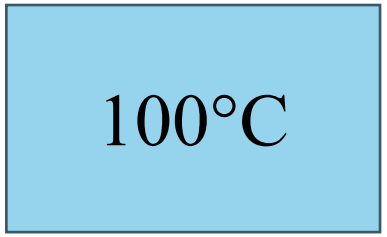
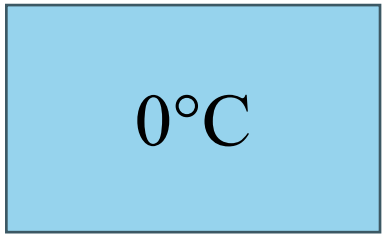
College of William & Mary, Williamsburg VA

Start with two equal masses m of water:



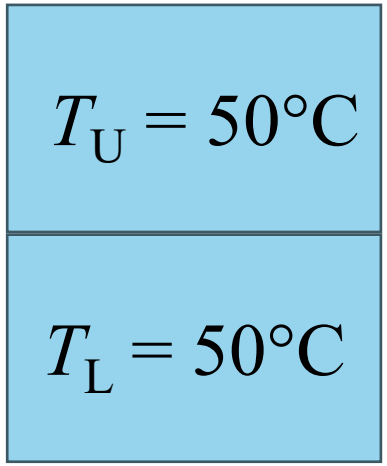
upper sample (U)



lower sample (L)

Call this step $n = 0$ in a process, with $T_U = 100°C$ and $T_L = 0°C$.

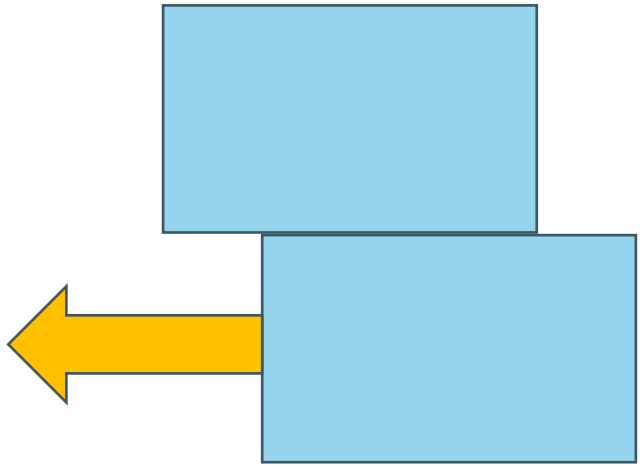
Step $n = 1$: Bring these samples into thermal contact and let them re-equilibrate.



What can we do differently if we want to end up
with a smaller value for T_U and
a larger value for T_L than 50°C
after contacting and then separating the samples?

(Assume that the specific heat c of water
is 4.186 J/g·°C independent of temperature.)

Instead of bringing the samples into contact transversely,
bring them into contact longitudinally.
In other words, slide one across the other.

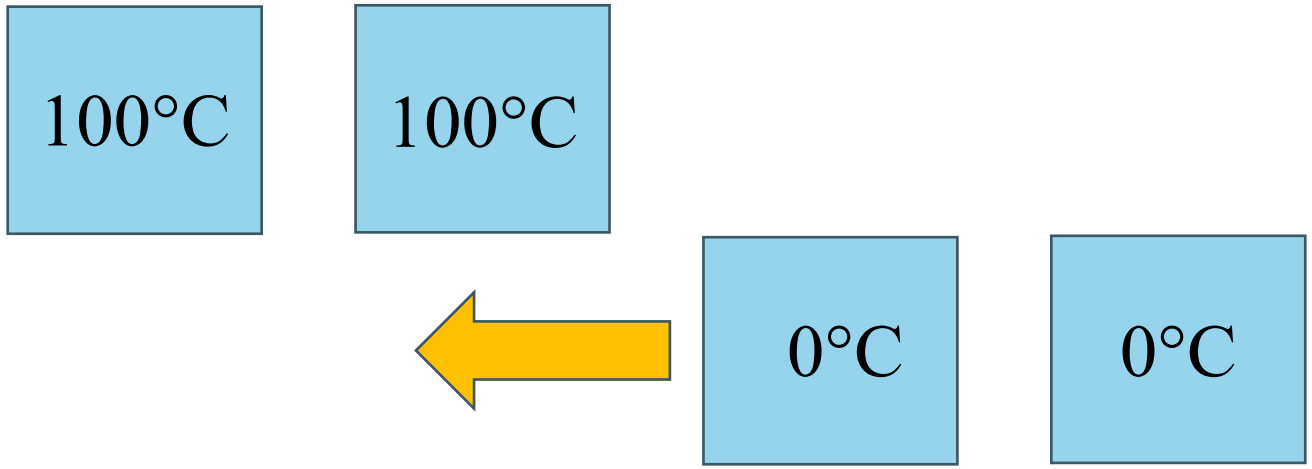


That works in principle.
But in practice the details would then depend on
factors such as the relative speed of sliding
and the thermal conductivity.

An equivalent but simpler approach is to
divide each sample into equal sub-samples
which will successively be brought in contact.

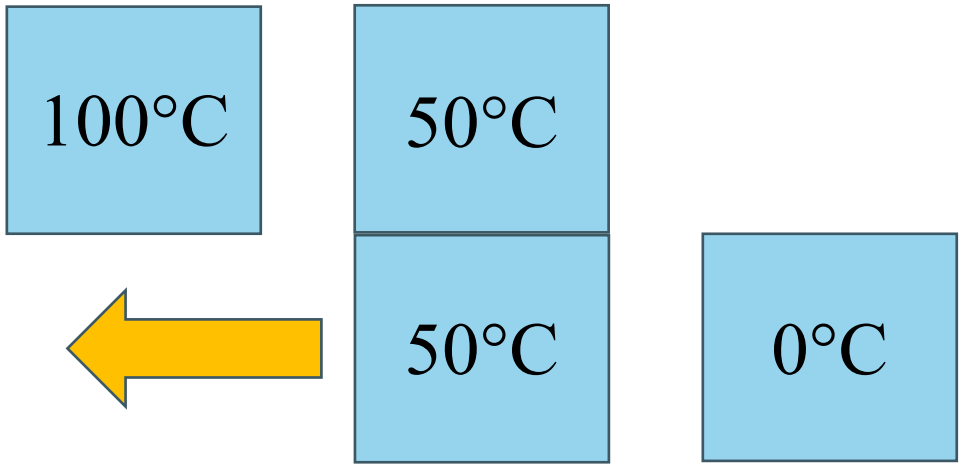
Let n be the number of sub-samples.
We already did the case of $n = 1$.

$n = 2$: initial configuration

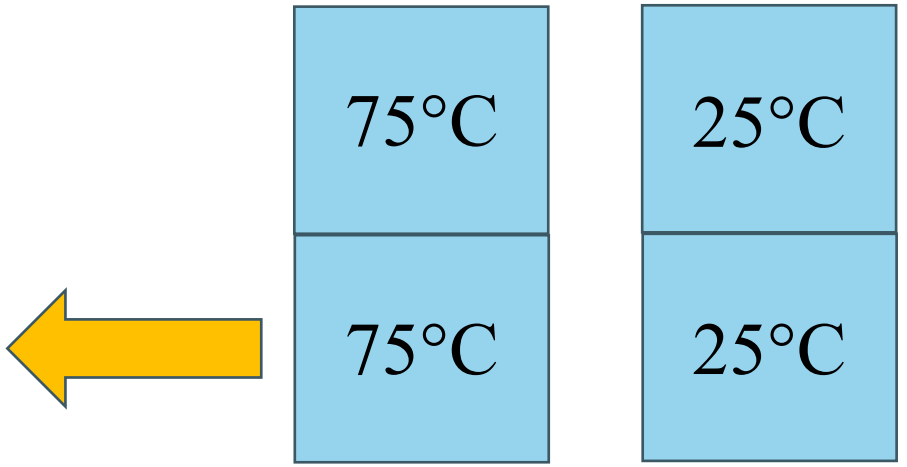


Move the lower train of sub-samples
past the upper train,
starting and stopping at each contact.

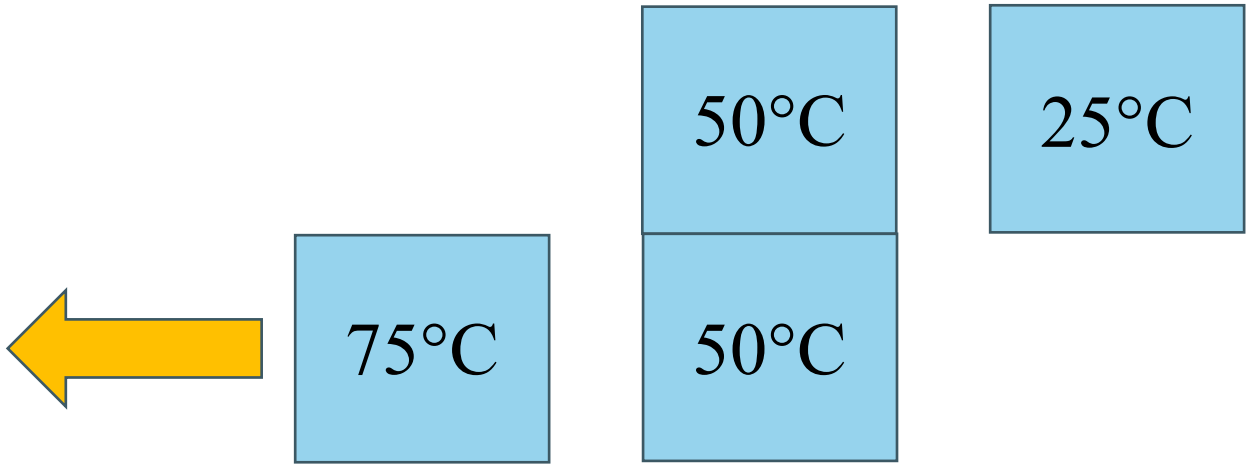
$n = 2$: first contact between sub-samples



$n = 2$: second contact



$n = 2$: third (and final) contact



$n = 2$: recombine the sub-samples

$T_U =$ 37.5°C average of 50°C & 25°C

$T_L =$ 62.5°C average of 75°C & 50°C

$n = 2$: final values of $T_U = 37.5°C$ & $T_L = 62.5°C$

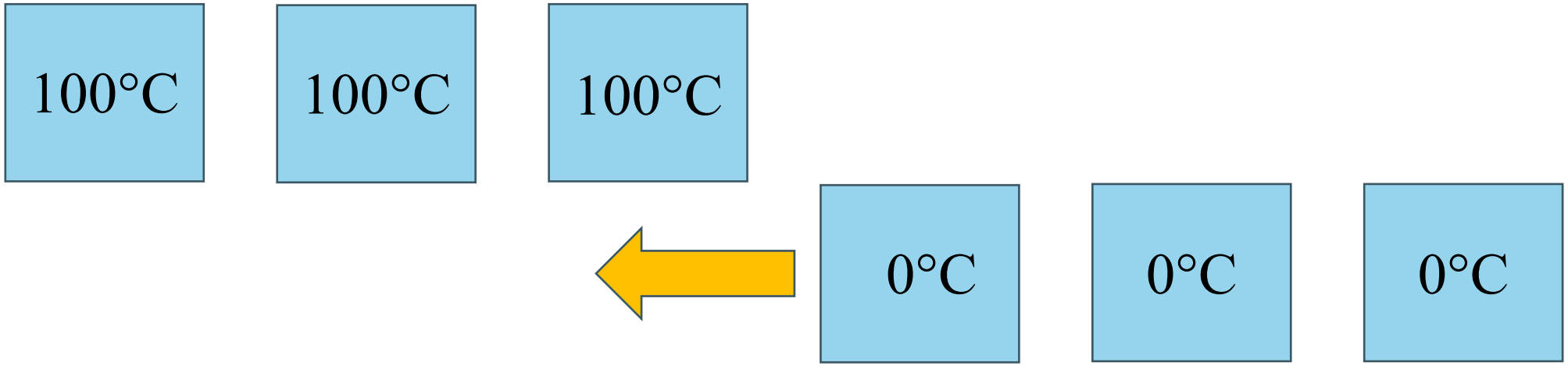
We indeed decreased T_U and
increased T_L from 50°C!

Two checks on thermal energy conservation:
(1) average of T_U and T_L is still 50°C
(2) sum of T_U and T_L is still 100°C

You can probably guess how we can
further this trend:

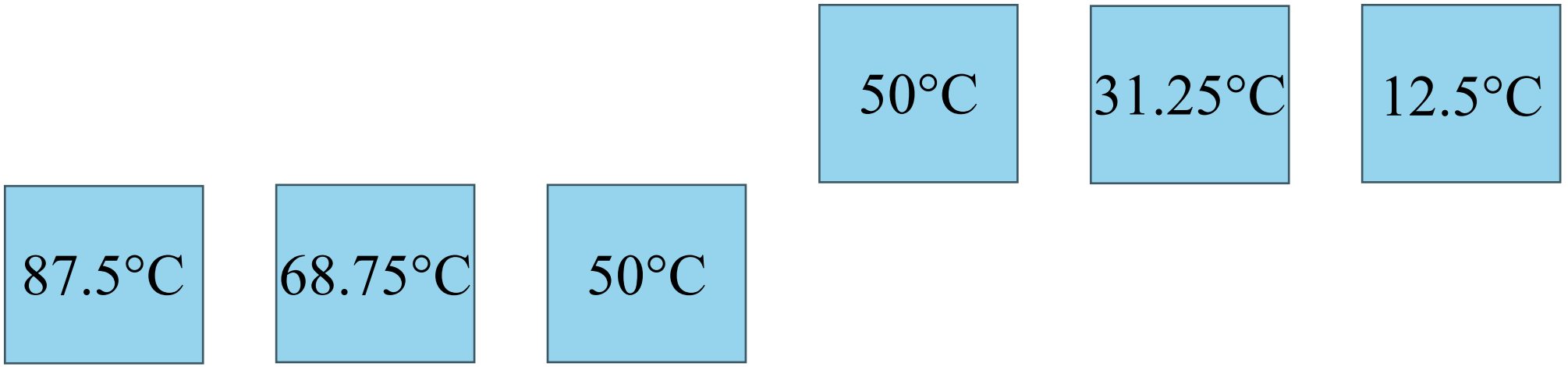
use $n = 3$ sub-samples.

$n = 3$: initial configuration



You can check the intermediate steps,
but let's just jump to the end.

$n = 3$: after final (fifth) contact



$n = 3$: recombine the sub-samples

$T_U =$ 31.25°C

$T_L =$ 68.75°C

Better than 37.5°C and 62.5°C for $n = 2$.

try $n = 5$: final temperatures of lower sub-samples from left to right

$$T_{L1} = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}\right)100°C$$

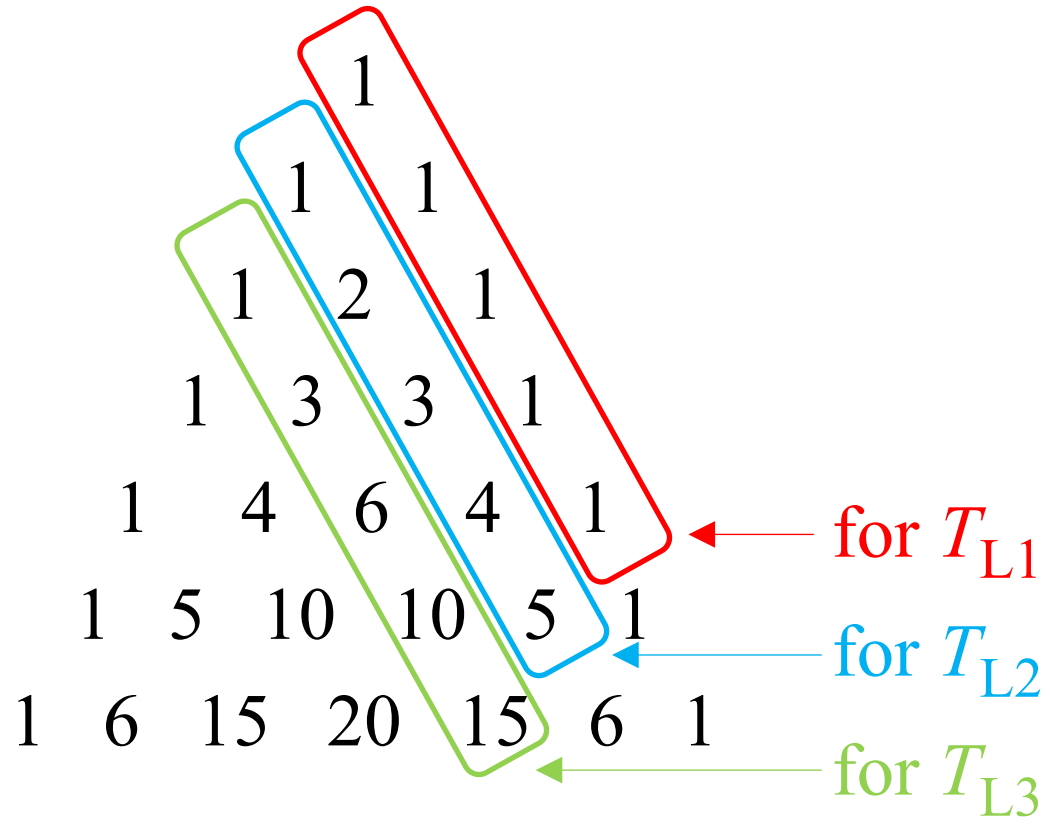
$$T_{L2} = \left(\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64}\right)100°C$$

$$T_{L3} = \left(\frac{1}{8} + \frac{3}{16} + \frac{6}{32} + \frac{10}{64} + \frac{15}{128}\right)100°C$$

and similarly for T_{L4} and T_{L5}

The denominators are successive powers of 2.
Do you recognize the numerators?

They are binomial coefficients,
given by the diagonals of Pascal's triangle:



Sum those terms to get the final recombined temperatures:

$$T_{\mathrm{U}} = \underbrace{\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdots\right)}_{n \text{ terms}} 100°\mathrm{C}$$

$$T_{\mathrm{L}} = 100°\mathrm{C} - T_{\mathrm{U}}$$

e.g. $n = 1 \quad \Rightarrow \quad T_{\mathrm{U}} = \frac{1}{2} 100°\mathrm{C} = 50°\mathrm{C}$

$n = 2 \quad \Rightarrow \quad T_{\mathrm{U}} = \frac{1}{2}\cdot\frac{3}{4} 100°\mathrm{C} = 37.5°\mathrm{C}$

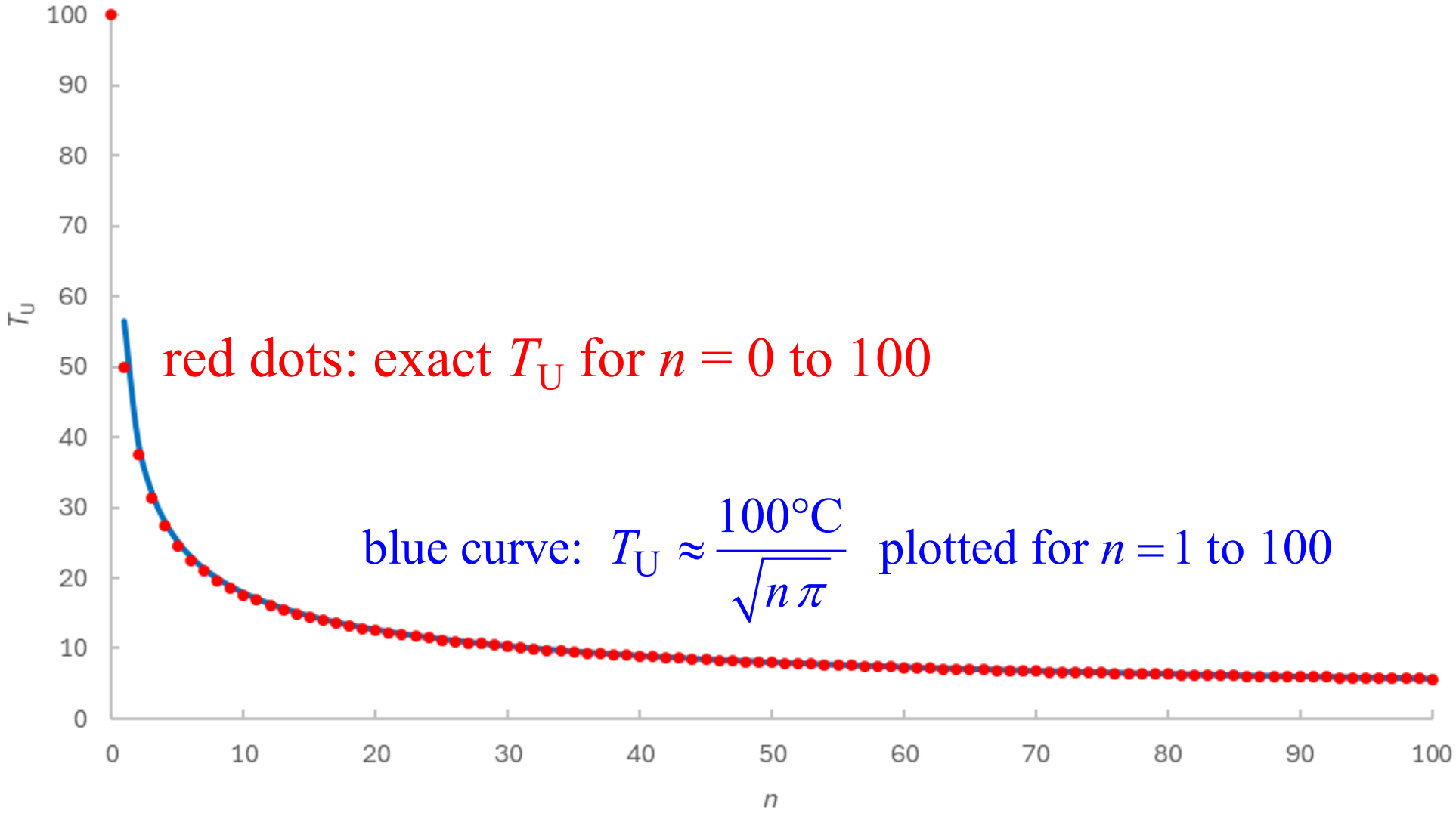
$n = 3 \quad \Rightarrow \quad T_{\mathrm{U}} = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6} 100°\mathrm{C} = 31.25°\mathrm{C}$

Large n limit using the Stirling approximation:

$$T_{\mathrm{U}} \approx \frac{100^{\circ}\mathrm{C}}{\sqrt{n\,\pi}} \to 0^{\circ}\mathrm{C} \quad \text{as} \quad n \to \infty$$

so that we exchanged T_{U} and T_{L}!

This limiting process is evidently reversible:
reverse it by running the lower train of sub-samples
back rightward across the upper train.


100
90
80
70
60
50
40
30
20
10
0
T_U
0
10
20
30
40
50
60
70
80
90
100
n
red dots: exact T_U for $n = 0$ to 100
blue curve: $T_U \approx \frac{100°C}{\sqrt{n\,\pi}}$ plotted for $n = 1$ to 100

Entropy change between the initial and final configurations:

$$\boxed{\frac{\Delta S}{mc} = \ln\left(\frac{273 + T_{\mathrm{U}}}{373}\right) + \ln\left(\frac{373 - T_{\mathrm{U}}}{273}\right)}$$

$n = 1: \quad T_{\mathrm{U}} = 50 \quad \Rightarrow \quad \frac{\Delta S}{mc} = \ln\frac{323}{373} + \ln\frac{323}{273} = 0.146$ maximally irreversible

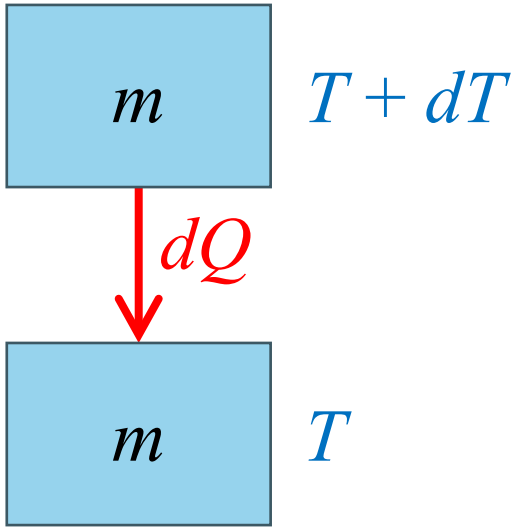
$n \to \infty: \quad T_{\mathrm{U}} = 0 \quad \Rightarrow \quad \frac{\Delta S}{mc} = \ln\frac{273}{373} + \ln\frac{373}{273} = 0$ reversible

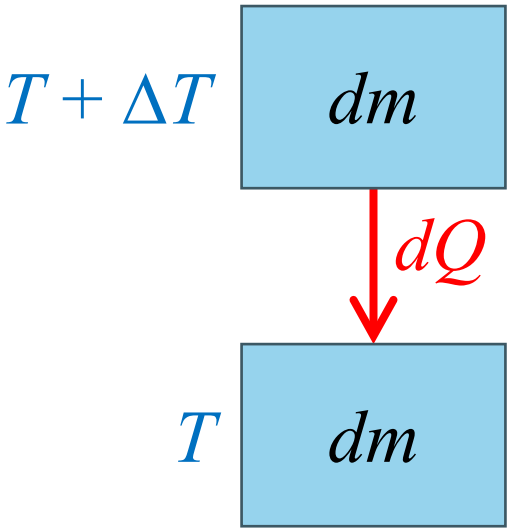
But how can the entropy change be zero
for heat exchange between two sub-samples
whose initial temperature difference ΔT
is <u>not</u> infinitesimal?

Answer: $dS = \dfrac{dQ}{T}$ is infinitesimal if dQ is infinitesimal!

The usual way to ensure dQ is infinitesimal:

$dQ = m\,c\,dT$ for two samples of infinitesimal temperature difference dT brought into contact.





An alternate way to ensure dQ is infinitesimal:

$dQ = dm\,c\,\Delta T$ for two samples of infinitesimal mass dm brought into contact.

REFERENCES

https://www.usna.edu/Users/physics/mungan/_files/documents/Scholarship/ReversibleHeatBetweenDifferentTemperatureObjects.pdf

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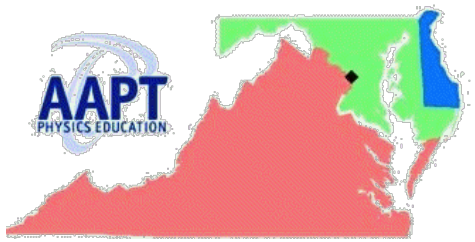
E.G. Mishchenko and P.F. Pschenichka, “Reversible temperature exchange upon thermal contact,” Am. J. Phys. **85**, 23–29 (Jan. 2017)



Comments or questions?



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