# Relativity on Rotated Graph Paper
# *Diagrams for the Muon Experiment*

**Rob Salgado**

**Hampton University**

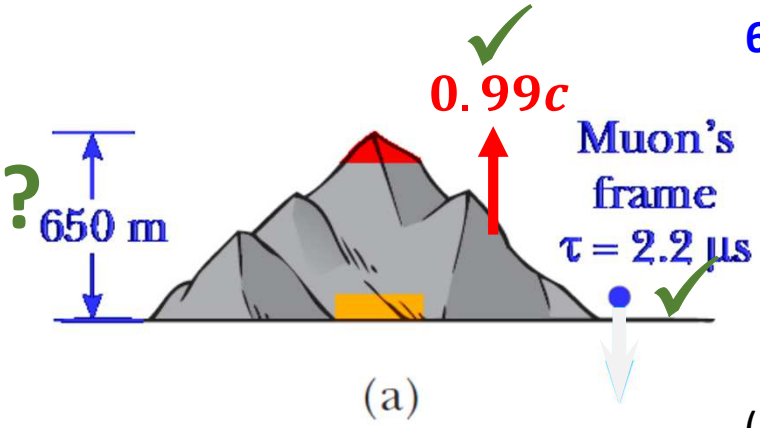
robertoBsalgado@gmail.com

The **Muon Experiment** problem in Special Relativity is treated using

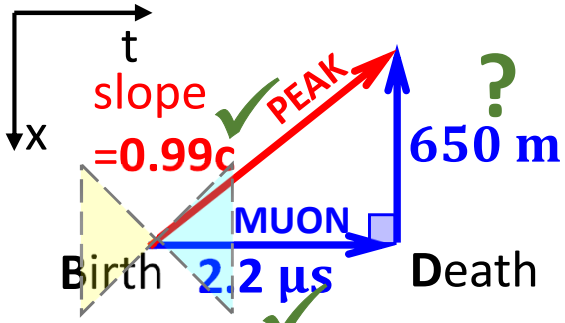
"**Relativity on Rotated Graph Paper"**
American Journal of Physics **84**, 344 (2016);
https://doi.org/10.1119/1.4943251

- The Muon Problem and its Spacetime Diagram
- **Visualize tickmarks** on a spacetime diagram,
  then **Calculate graphically**
  by **counting ticks** and doing **simple algebra**.
- First, use a **simpler numerical example** $(\boldsymbol{v = 0.8c,\ \tau = 6\ \mu s})$
  is constructed and interpreted.
- Then, use more **realistic values** $(\boldsymbol{v = 0.99c,\ \tau = 2.2\ \mu s})$
  to obtain the standard numerical results.
- With a fuller development of the method for students,
  this and other standard relativity problems
  can be used in an introductory ***algebra-based*** physics course.

# The **Muon** problem

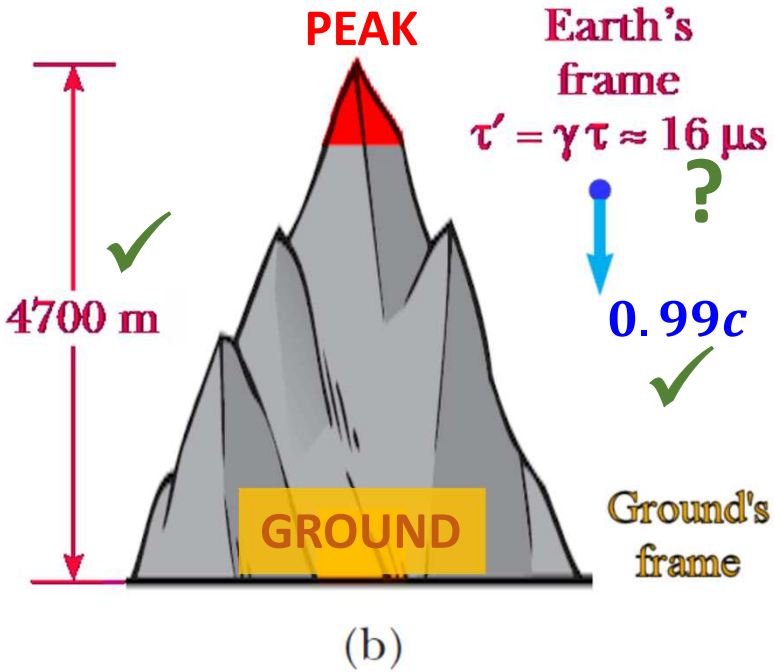


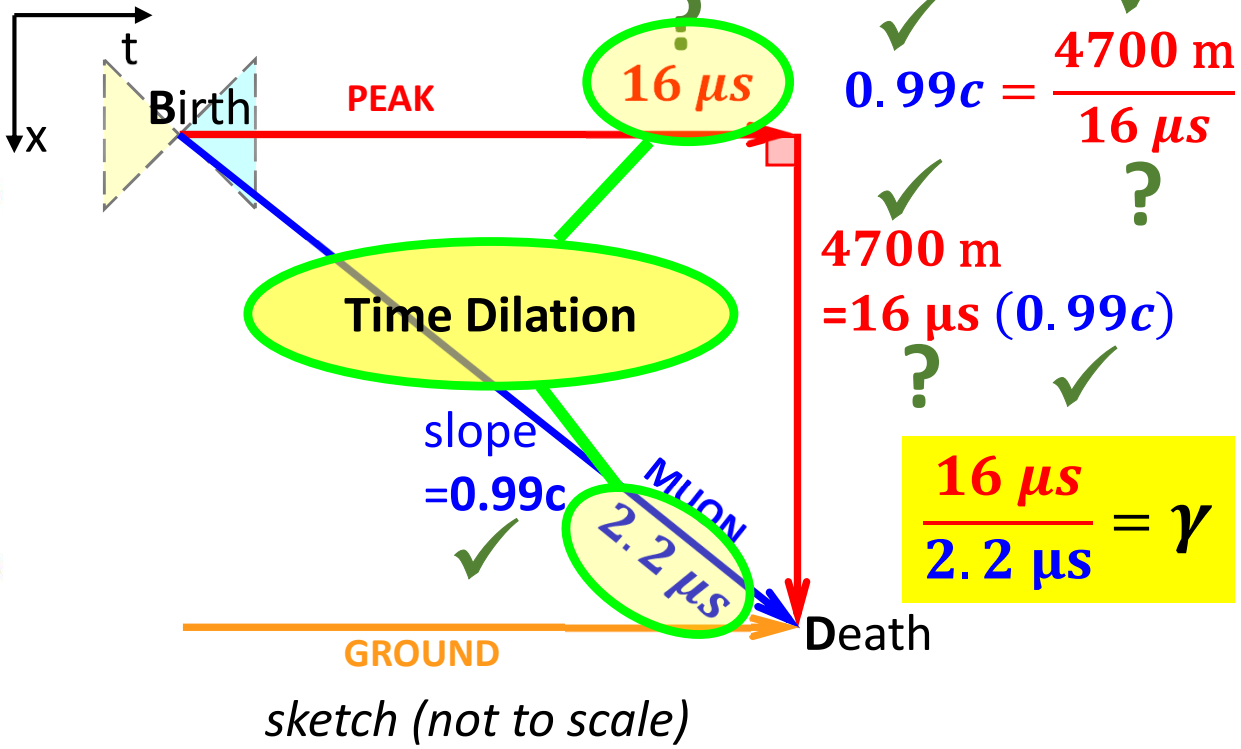

(a) **Muons** (whose lifetime is about **2.2μs**) observe the ***Earth*** (traveling with a speed of **0.99c**) travels only about **650m**, as measured in the **muons**' frame.




$$0.99c = \frac{650\ \text{m}}{2.2\ \mu\text{s}}$$

(b) An observer on **Earth** measures **muons** *(traveling with speed **0.99c**)* travel about **4700m**. Because of time dilation, the **Earth** observer measures the **muons**' lifetime to be **longer than 2.2μs**.



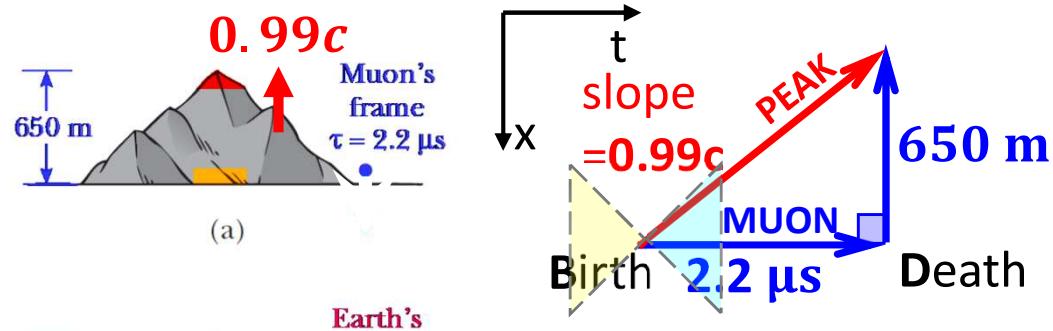

$$0.99c = \frac{4700\ \text{m}}{16\ \mu s}$$

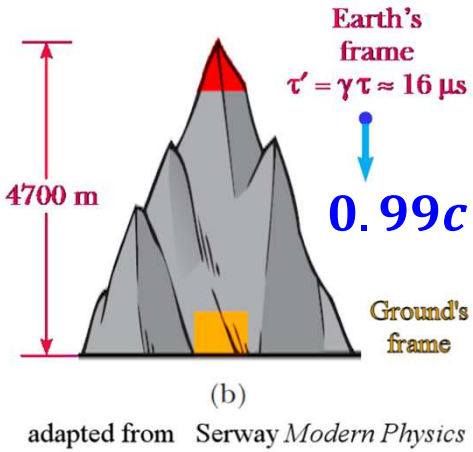
**4700 m**
**=16 μs (0.99c)**

$$\frac{16\ \mu s}{2.2\ \mu s} = \gamma$$

*sketch (not to scale)*

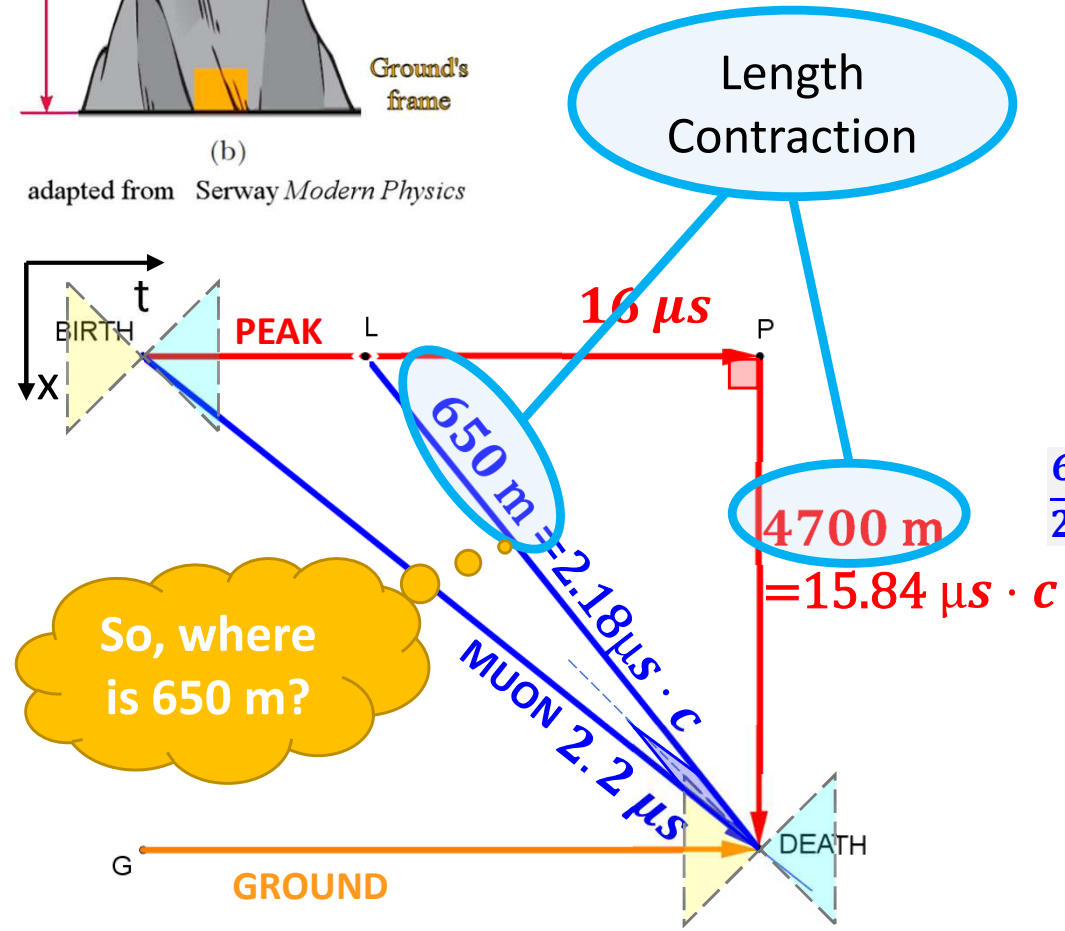
Fig 1.11 from Serway, *Modern Physics (3e)* uses the given approximate values.




# Need better intuition about what is going on.

- Our PHY 101 use of position-vs-time graphs is *[unfortunately] only an approximation.*
- It's **not Euclidean geometry.**
  … the Pythagorean theorem doesn't work!
  … those blue-legs are legs of a right-triangle!
- Instead, **it's Spacetime geometry!**



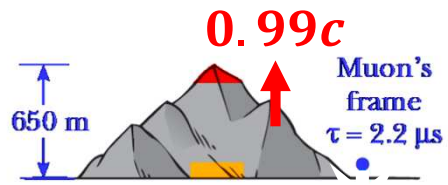

*sketch (not to scale)*

$$\frac{650\ \text{m}}{2.2\ \mu\text{s}} = 0.99c = \frac{4700\ \text{m}}{16\ \mu\text{s}}$$

$$\frac{16\ \mu\text{s}}{2.2\ \mu\text{s}} = \gamma = \frac{4700\ \text{m}}{650\ \text{m}}$$

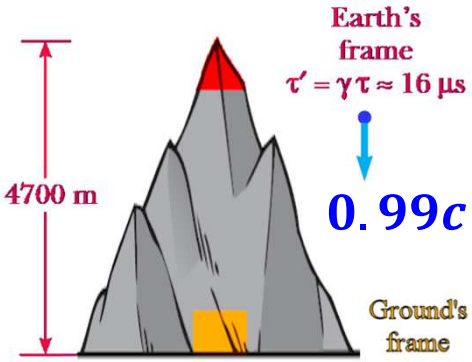
$$\frac{OPP}{ADJ} = c \cdot TANH\,\theta = v$$

$$\frac{ADJ}{HYP} = COSH\,\theta = \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$



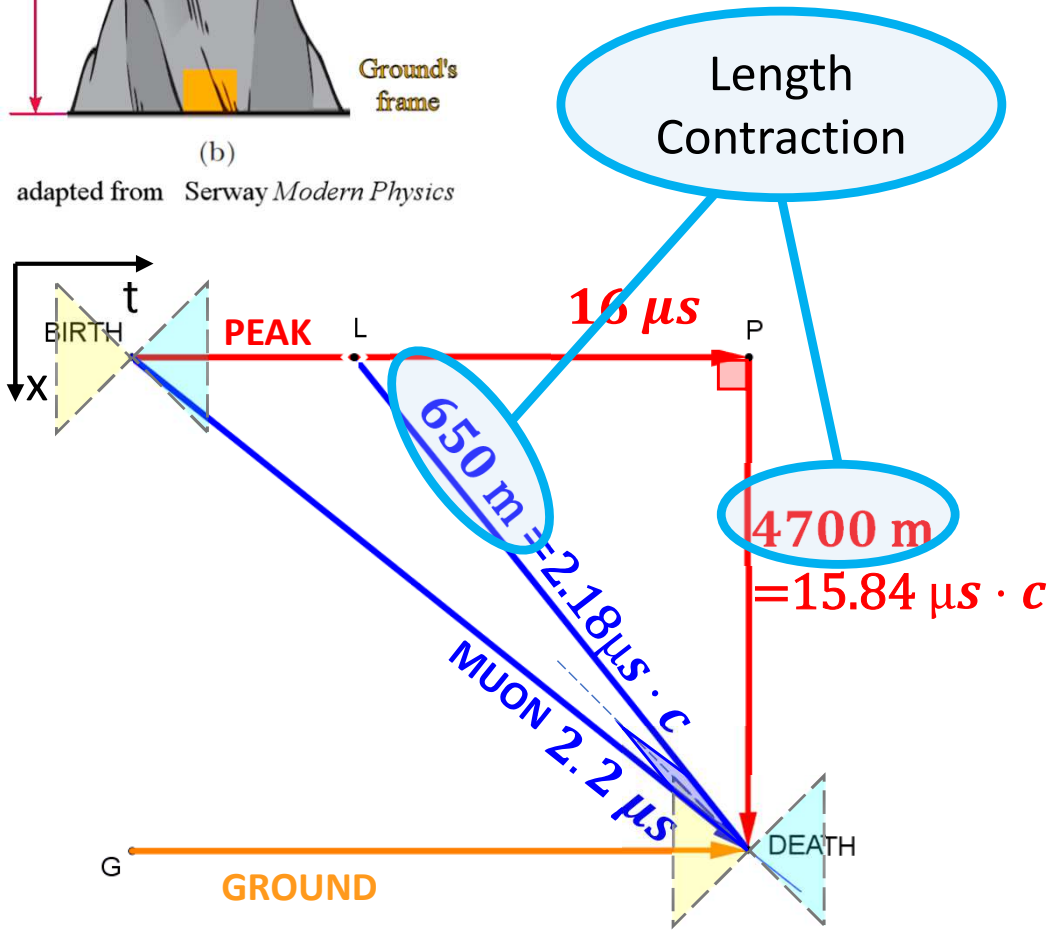

(a)

$$\frac{650\ \text{m}}{2.2\ \mu s} = 0.99c = \frac{4700\ \text{m}}{16\ \mu s}$$




(b)

adapted from Serway *Modern Physics*




*sketch (not to scale)*

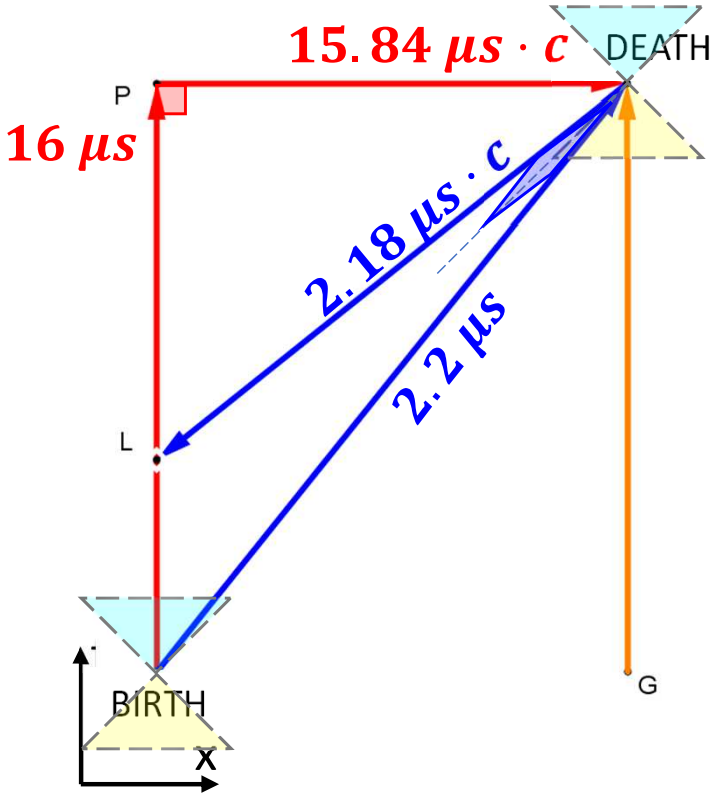
# Need better intuition about what is going on.

Arithmetic is easier with light-seconds.

$$\frac{2.18\ \mu s \cdot c}{2.2\ \mu s} = 0.99c = \frac{15.84\ \mu s \cdot c}{16\ \mu s}$$

$$\frac{16\ \mu s}{2.2\ \mu s} = \gamma = \frac{15.84\ \mu s \cdot c}{2.18\ \mu s \cdot c}$$

Relativity convention: *time-runs-upwards*



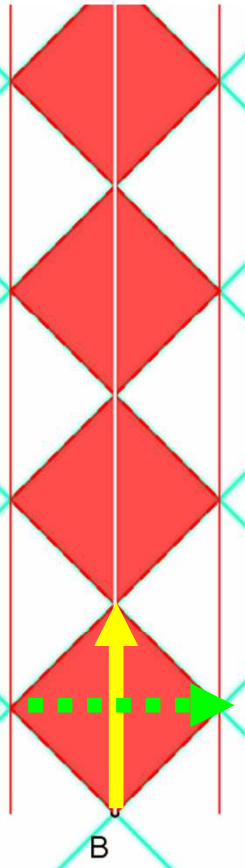
# spacetime diagrams of "Longitudinal Light Clocks"

## Light Clock Diamonds as "ticks of time"




rear mirror

front mirror

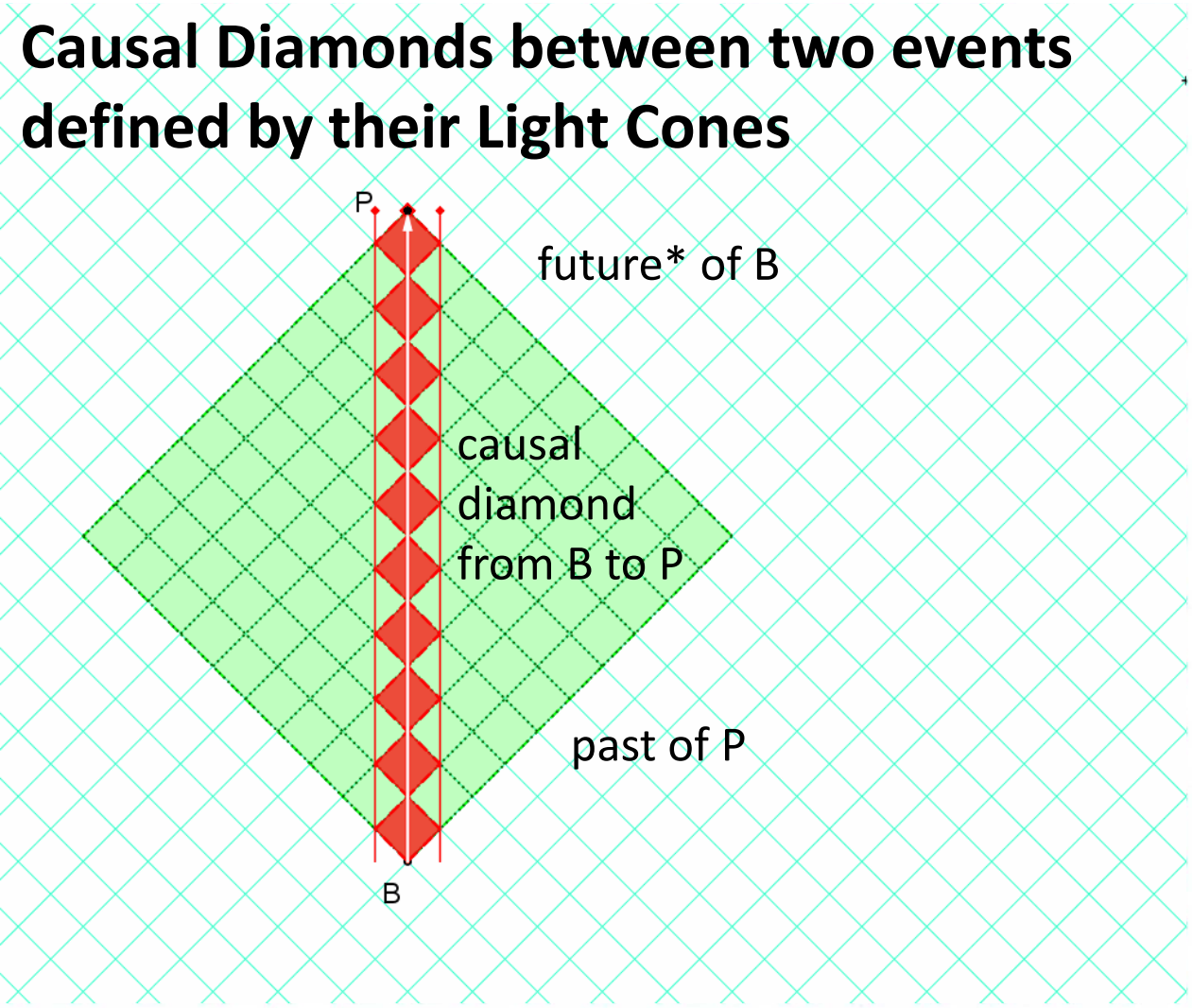



**1 tick (along the timelike diagonal)**

**1 space-tick = 1 *stick* (along the spacelike diagonal)**

**Rotated Graph Paper will be useful for visualizing measurements in spacetime!**

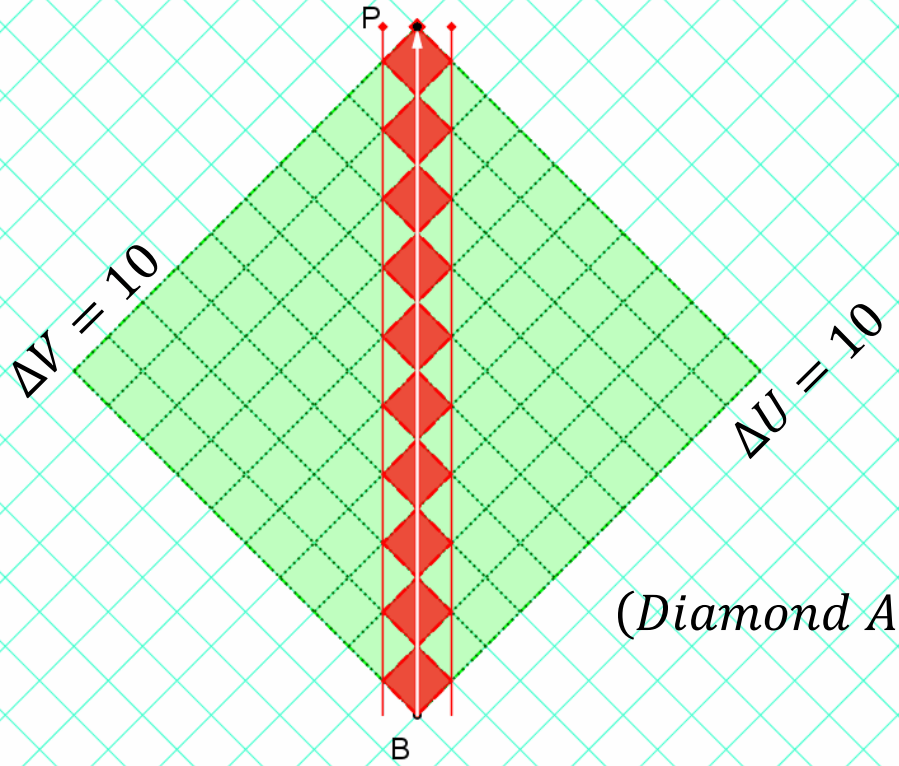
# Causal Diamonds between two events defined by their Light Cones




* the interior of the future light cone of B

**Causal Diamond:**

$(diamond\ area) = (square\text{–}interval)$

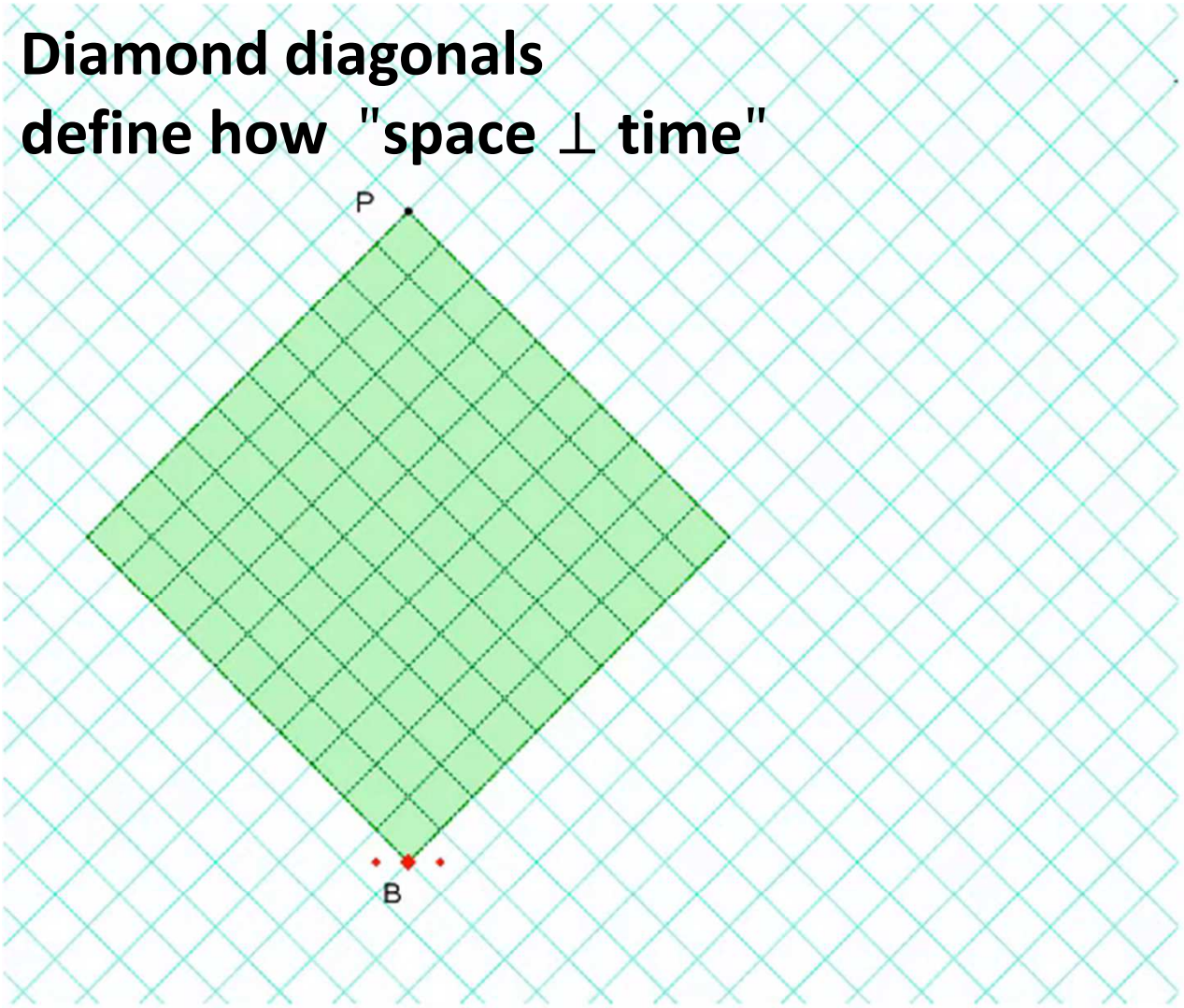


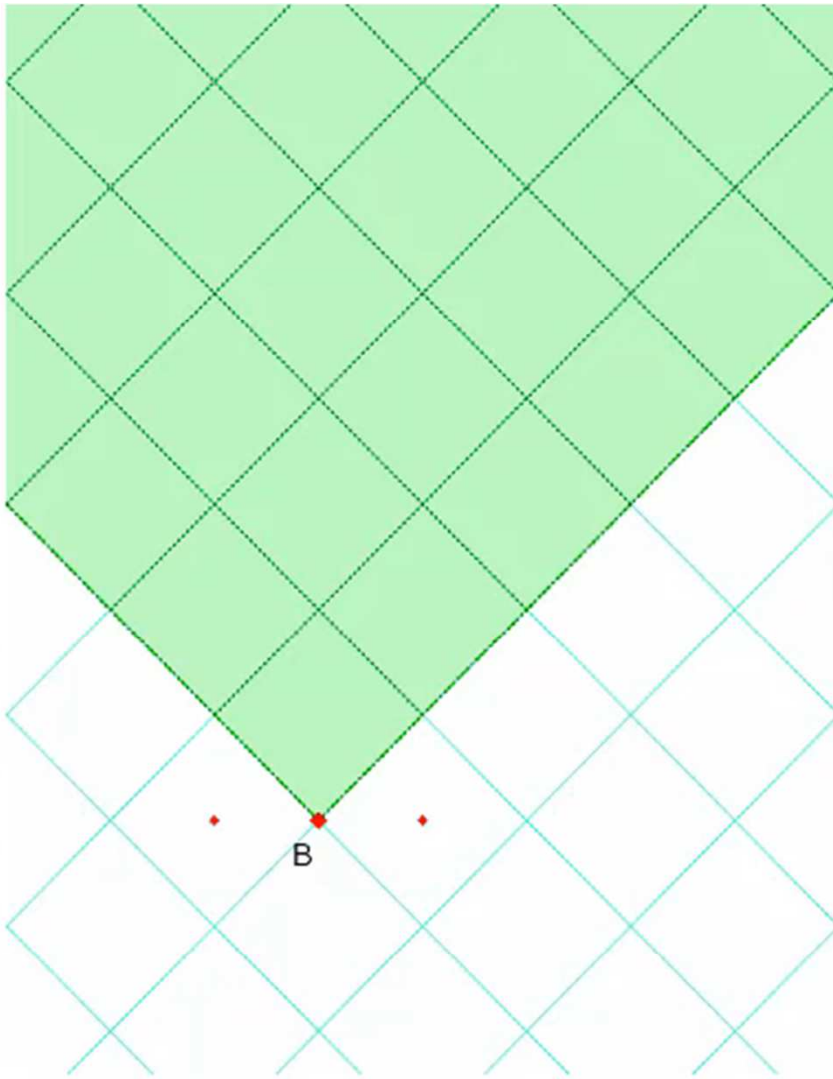

$$(Diamond\ Area) = \Delta U\ \Delta V$$
$$= (10)(10) = 100 = (diagonal)^2$$
$$= (10)^2$$

$$(\textbf{\textit{number of ticks along diagonal}}) = \sqrt{\begin{pmatrix} \textbf{\textit{Area}}\ of\ its\ causal\ diamond \\ in\ units\ of\ light\ clock\ diamonds \end{pmatrix}}$$

$$\Delta\tau_{BP} = \sqrt{(Area\ of\ causal\ diamond\ BP)}$$

# Diamond diagonals define how "space ⊥ time"

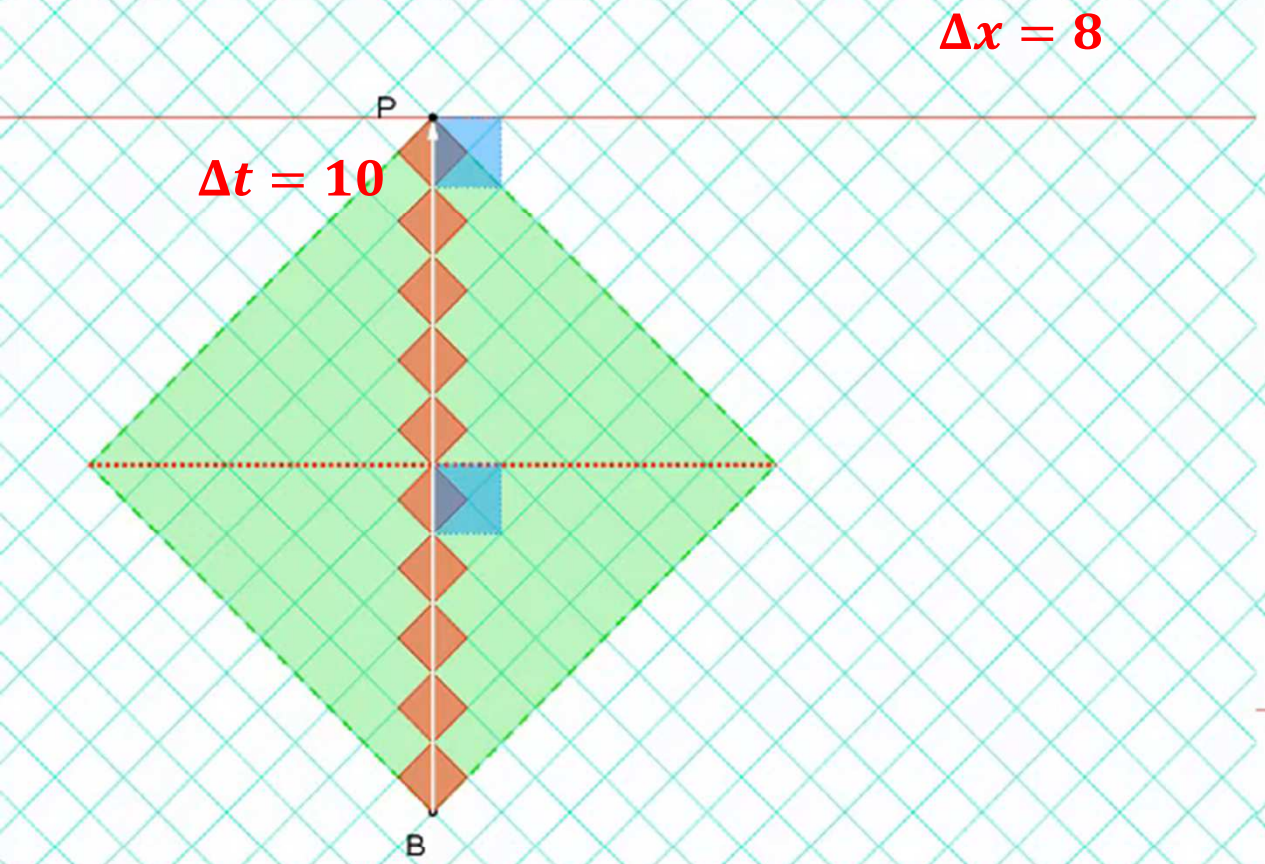


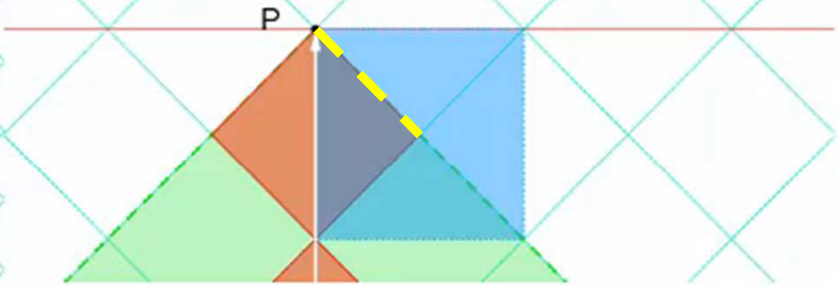

"A pair of events that are **simultaneous** for the inertial observer along BP"
lie on lines parallel to the
spacelike diagonal of the diamond BP.

The spacelike diagonal (the spaceline)
is **perpendicular** to the timelike diagonal (the worldline [or, better, timeline]).
This captures the notion: "**space ⊥ time**".

# Draw the spatial displacement from P



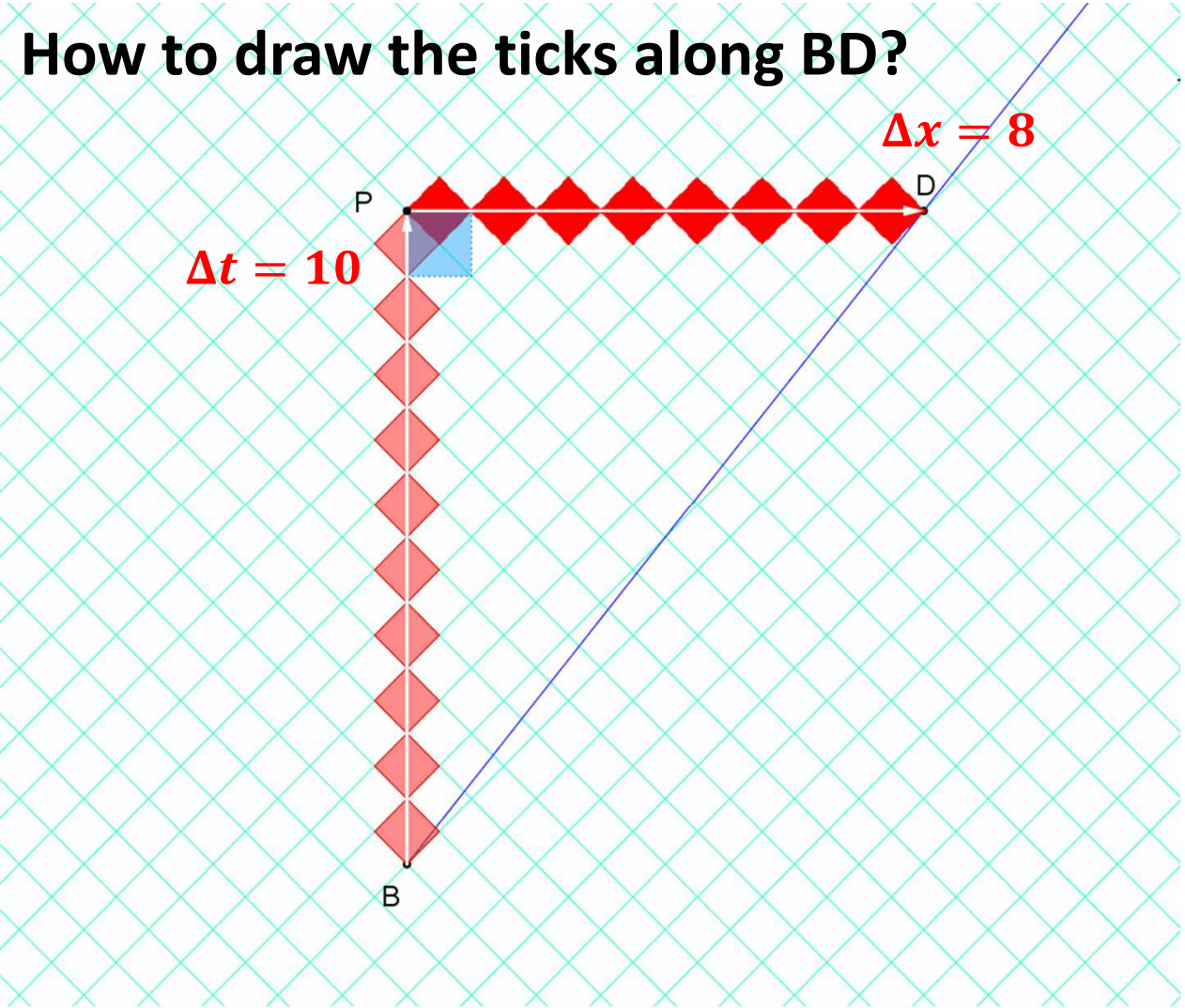

The diamonds
for a tick and for a stick
*share an edge* along the grid.




We can now measure off a **spatial distance perpendicular to the worldline.**

For (4/5)c, go out 8 sticks for every 10 ticks.

# How to draw the ticks along BD?

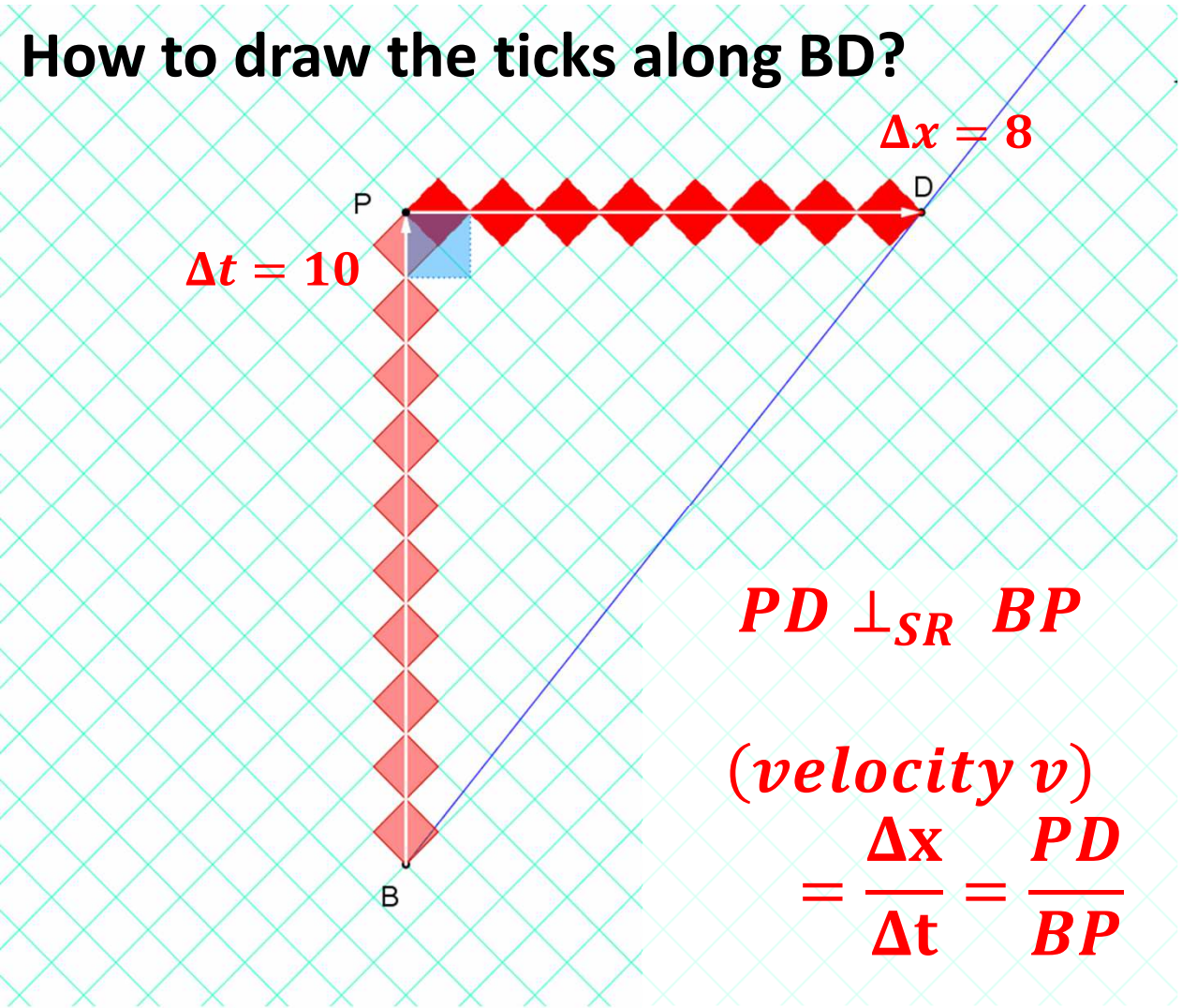



$$PD \perp_{SR} BP$$

$$(velocity\ v) = \frac{\Delta x}{\Delta t} = \frac{PD}{BP}$$

For (4/5)c, go out 8 sticks for every 10 ticks.
So, BD traces the worldline of our fictional muon.
**Q: How are the ticks of time drawn along *that worldline?***

# How to draw the ticks along BD?




**Q: How are the ticks of time drawn along *that worldline?***

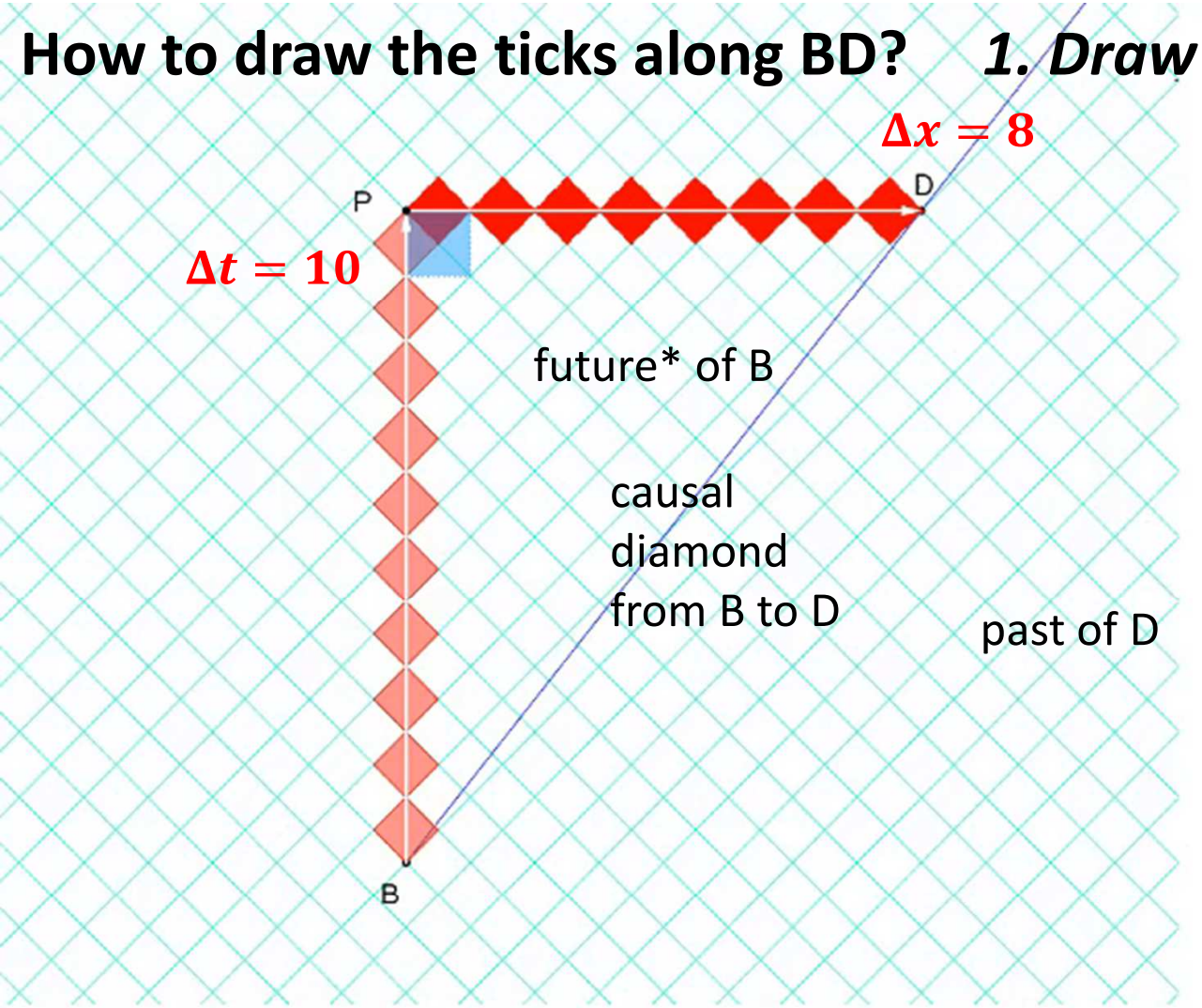
**Calibrating Bob's Longitudinal Light Clock**

geogebra.org/m/kvfsq664

**Hyperbola Pairs 2019 [2021 update]**

desmos.com/calculator/wtwnyljbzy

## How to draw the ticks along BD? *1. Draw the causal diamond of BD!*



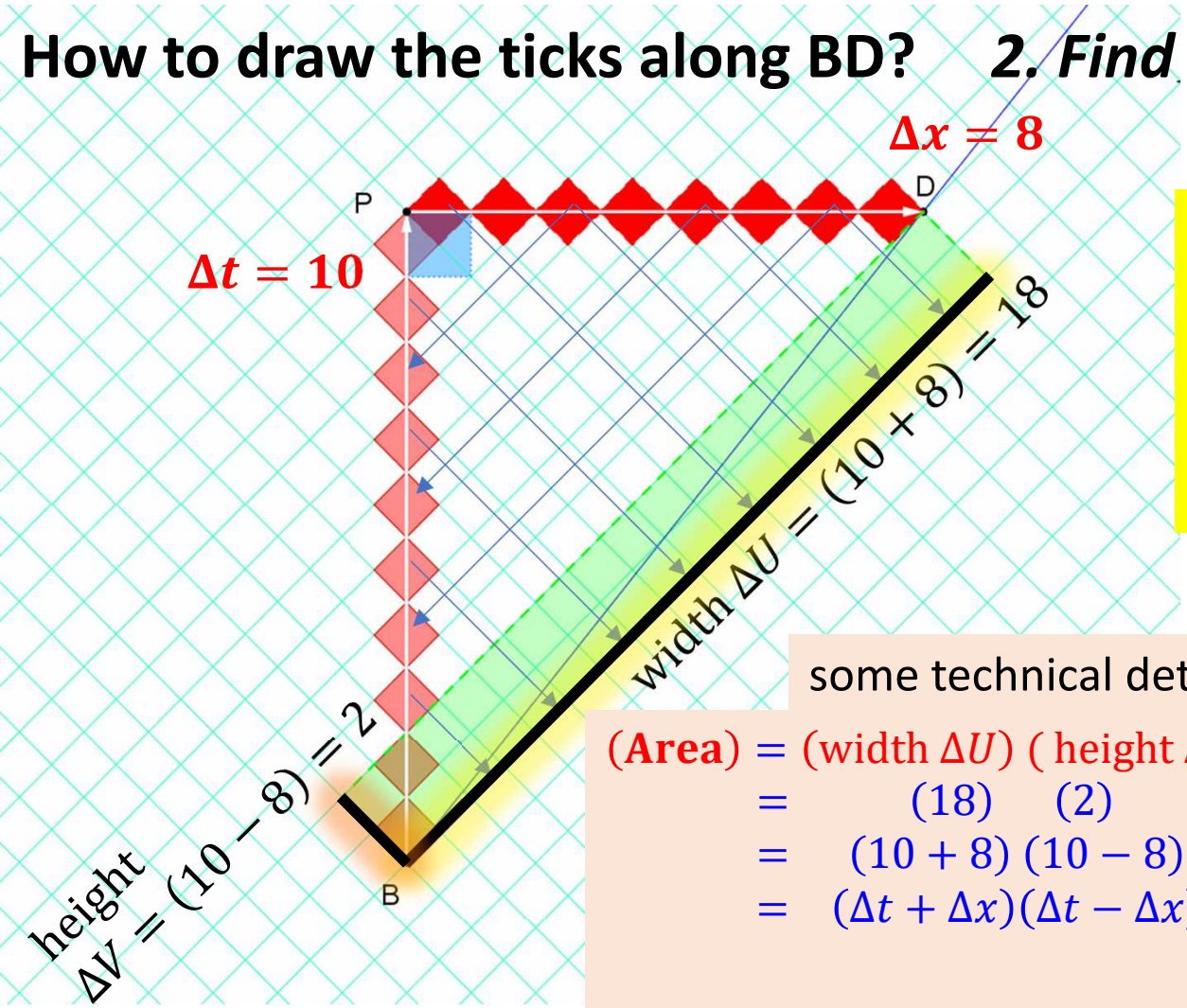

**Q: How are the ticks of time drawn along *that worldline?***

A: Draw the causal diamond of BD.

Then ($\boldsymbol{number\ of\ ticks\ along\ diagonal}$)

$$= \sqrt{\left(\begin{matrix}\boldsymbol{Area}\ of\ its\ causal\ diamond \\ in\ units\ of\ light\ clock\ diamonds\end{matrix}\right)}$$

# How to draw the ticks along BD? *2. Find the area of that diamond.*



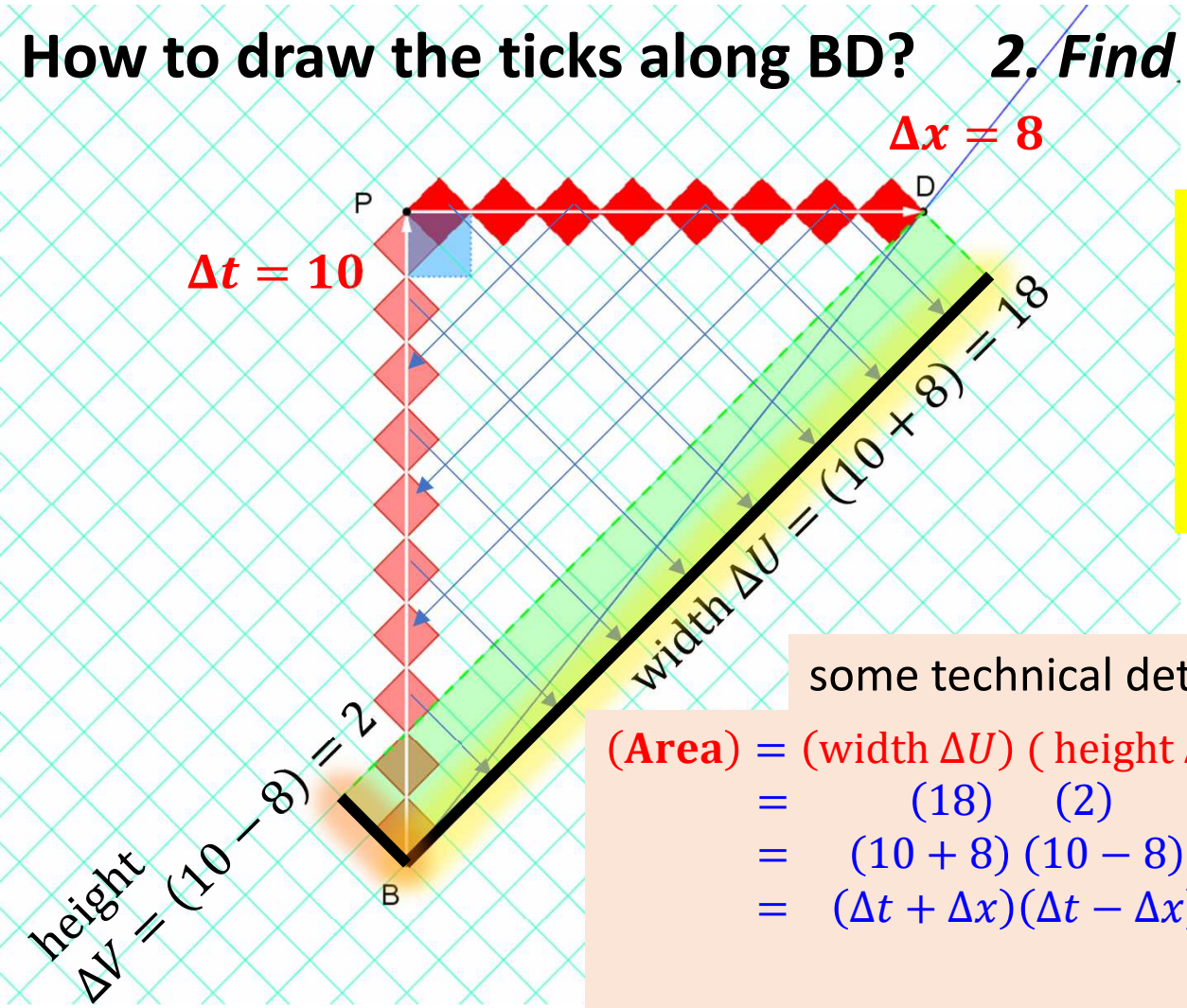

Take-away message

$$\Delta\tau_{BD} = \sqrt{Area}$$
$$= \sqrt{(18)(2)} = 6$$

some technical details (from my article)

$$(\mathbf{Area}) = (\text{width } \Delta U)\ (\text{height } \Delta V)$$
$$= \quad (18) \quad (2) \qquad\qquad = 36$$
$$= \quad (10+8)\,(10-8) \quad = 10^2 - 8^2 \quad = (6)^2$$
$$= \quad (\Delta t + \Delta x)(\Delta t - \Delta x) = \Delta t^2 - \Delta x^2$$
$$= (\quad \Delta\tau^2 \quad) = \begin{pmatrix} \textbf{\textit{Square interval}} \\ \textbf{\textit{of diagonal}} \end{pmatrix}$$

# How to draw the ticks along BD? *2. Find the area of that diamond.*




Take-away message

$$\Delta\tau_{BD} = \sqrt{Area}$$
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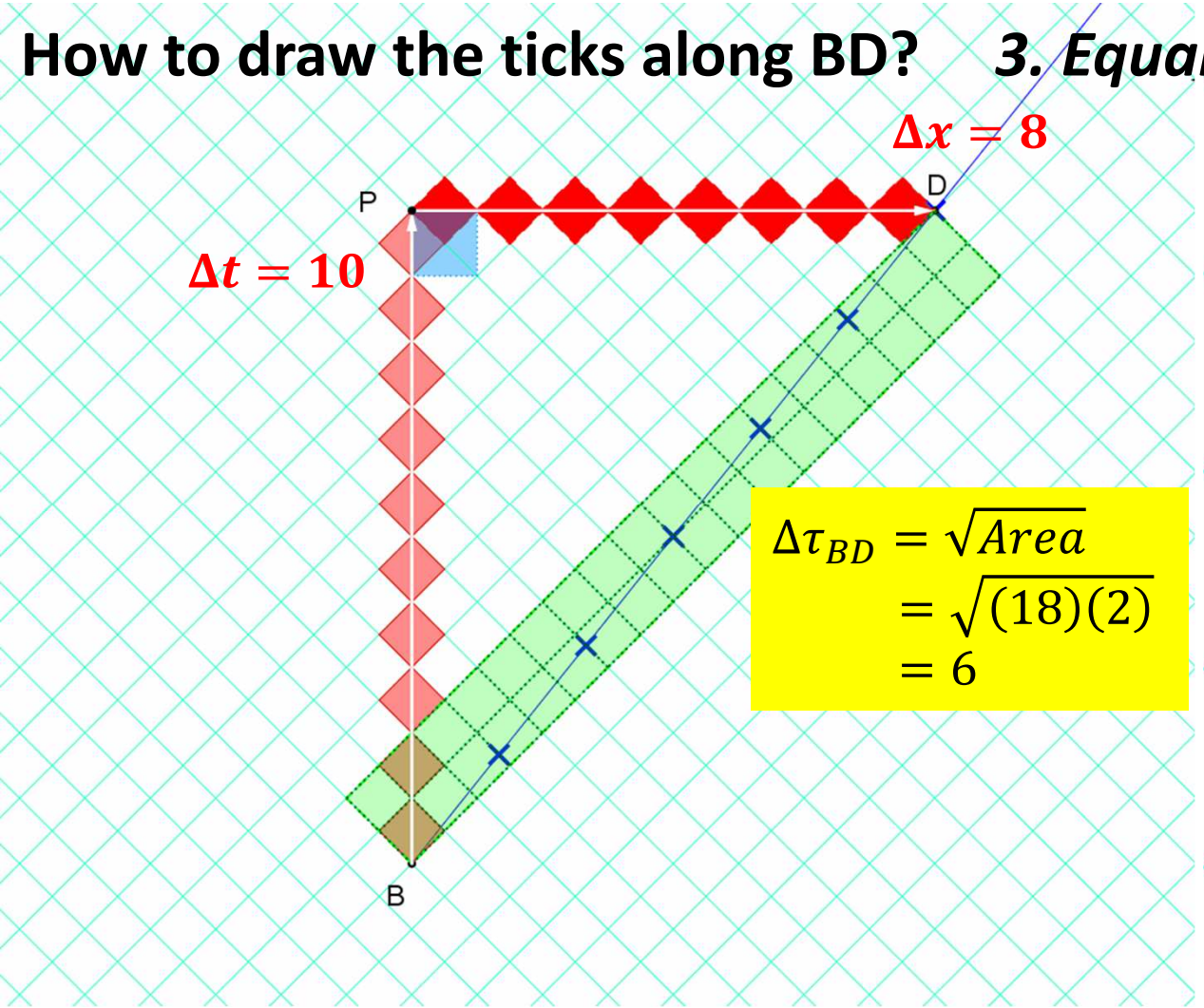
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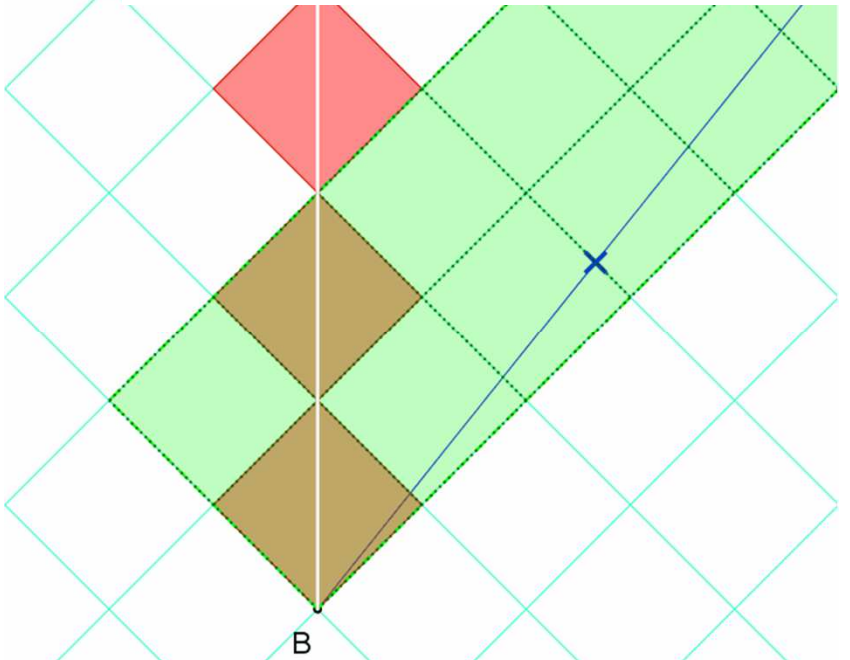
$$\textbf{(Velocity)} = \left(\frac{\Delta x}{\Delta t}\right) = \frac{(\Delta U - \Delta V)/2}{(\Delta U + \Delta V)/2} = \frac{\left(\frac{\Delta U}{\Delta V}\right)-1}{\left(\frac{\Delta U}{\Delta V}\right)+1} = \frac{\left(\begin{matrix}\textbf{Aspect}\\ \textbf{ratio}\end{matrix}\right)-1}{\left(\begin{matrix}\textbf{Aspect}\\ \textbf{ratio}\end{matrix}\right)+1}$$
$$= \frac{\left(\frac{18}{2}\right)-1}{\left(\frac{18}{2}\right)+1} = \frac{8}{10}$$

$$\left(\begin{matrix}\textbf{Aspect}\\ \textbf{ratio}\end{matrix}\right) = \left(\frac{\text{width } \Delta U}{\text{height } \Delta V}\right) = \left(\begin{matrix}square\ of\\ \textbf{Doppler factor}\end{matrix}\right) = k^2 = \left(\frac{18}{2}\right) = (3)^2$$

## How to draw the ticks along BD? *3. Equally divide the diagonal into $\sqrt{Area}$ parts.*

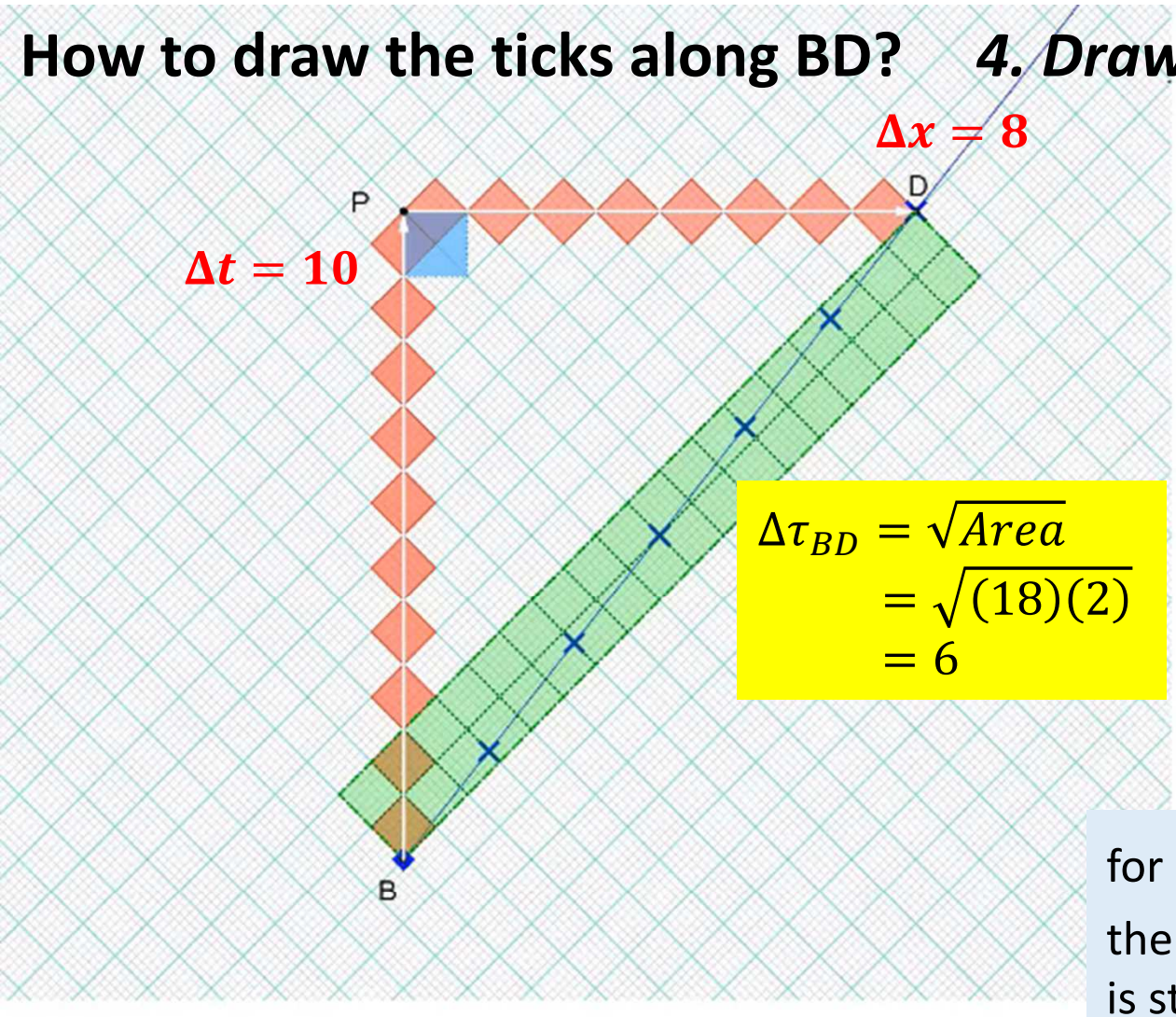



$$\Delta\tau_{BD} = \sqrt{Area} = \sqrt{(18)(2)} = 6$$

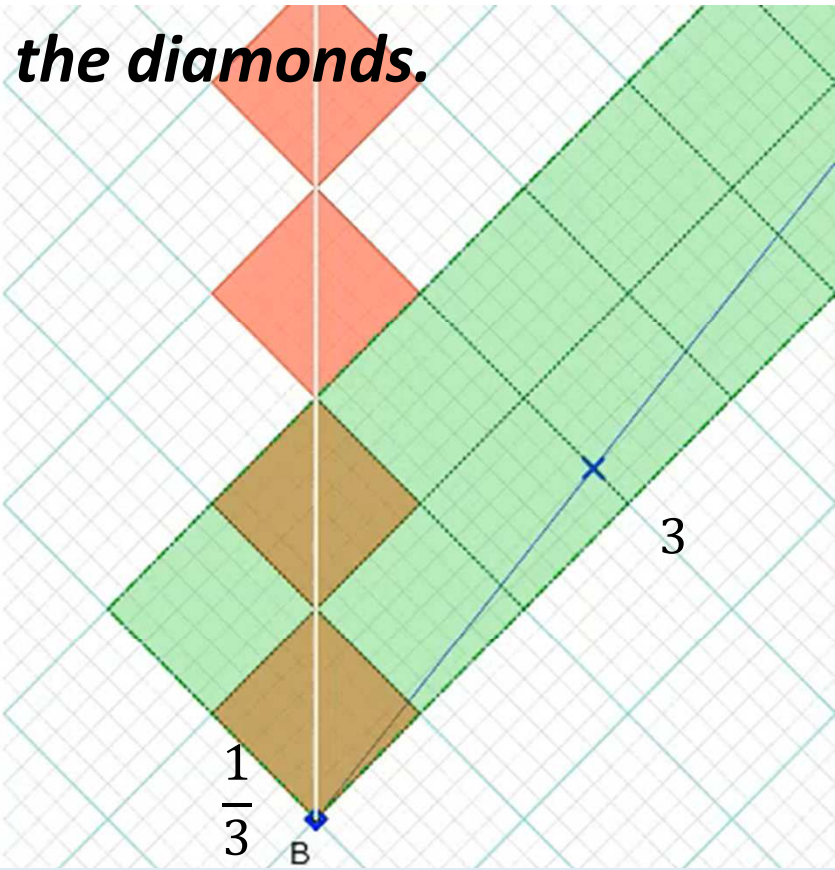



So, divide diagonal BD into $\sqrt{36} = 6$ equal parts.
(A 6x6 subgrid may help.)
The light clock diamonds are
geometrically similar to the causal diamond.

# How to draw the ticks along BD? *4. Draw the diamonds.*



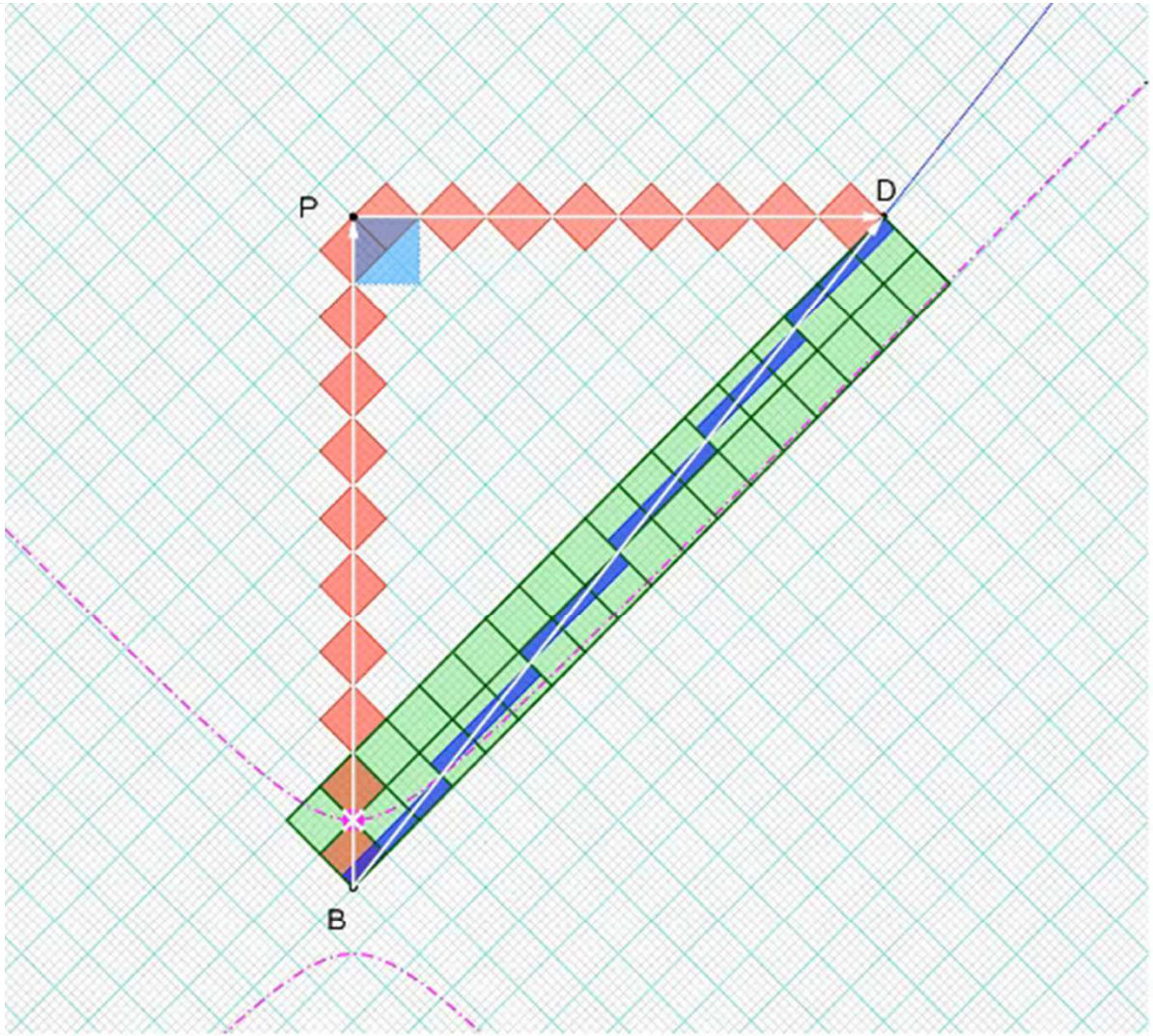

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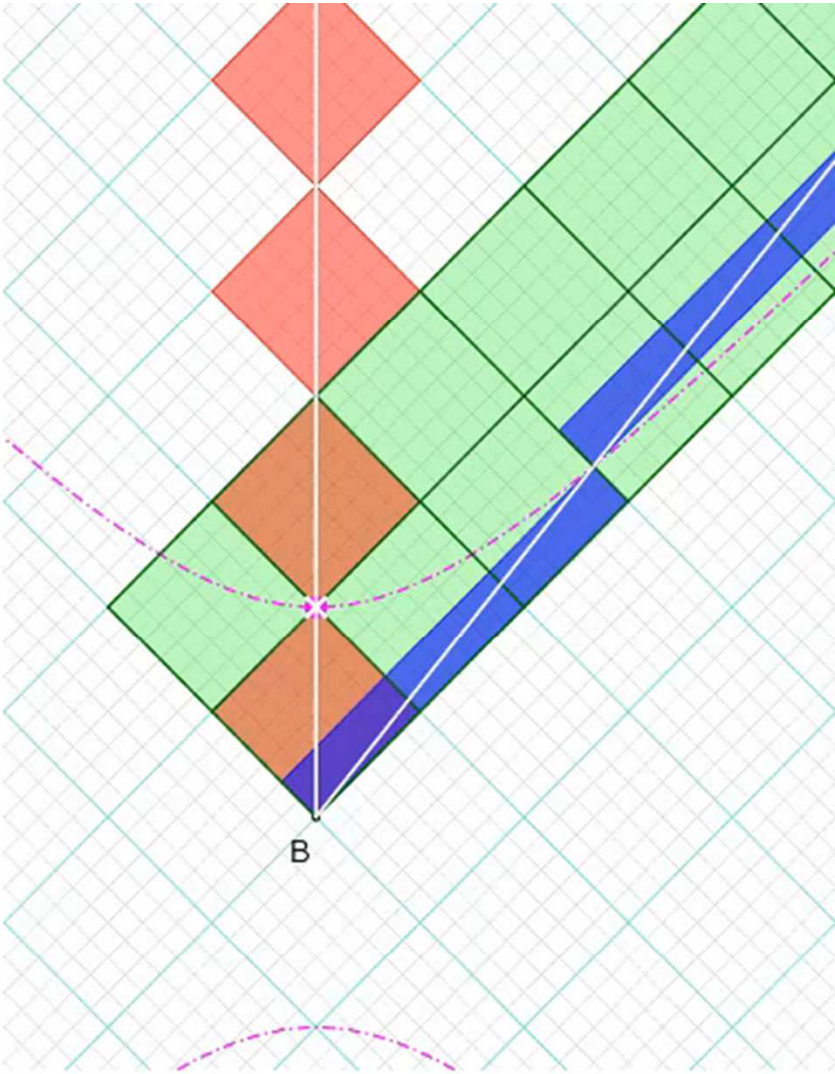



for $v = \left(\frac{4}{5}\right) c$,
the original clock diamond
is stretched by $k = 3$ along one direction
and shrunk by $k$ along the other direction
(which maintains the diamond area)

So, divide diagonal BD into $\sqrt{36} = 6$ equal parts.
(A 6x6 subgrid may help.)
The light clock diamonds are
geometrically similar to the causal diamond.

Let's trace them out!

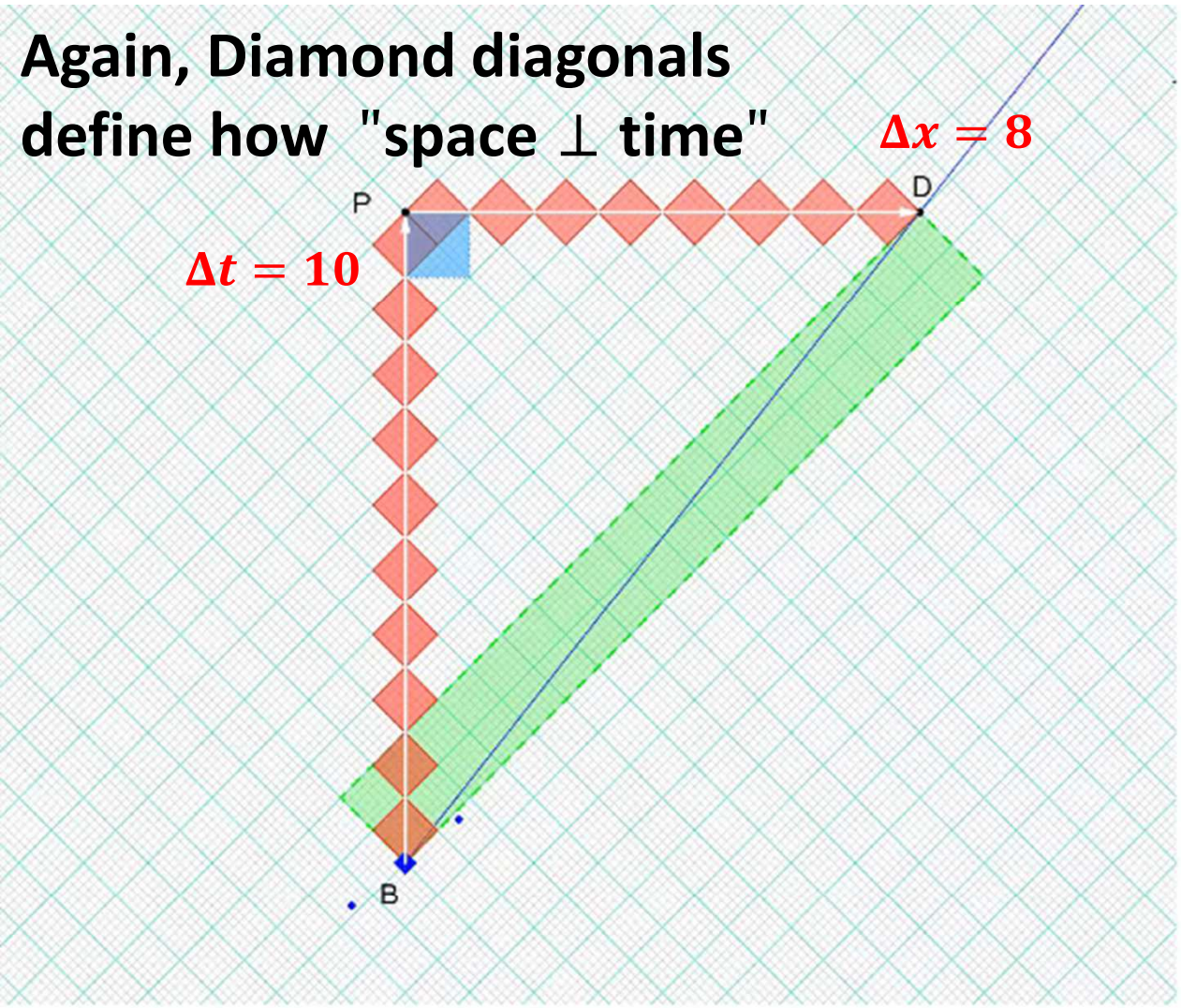


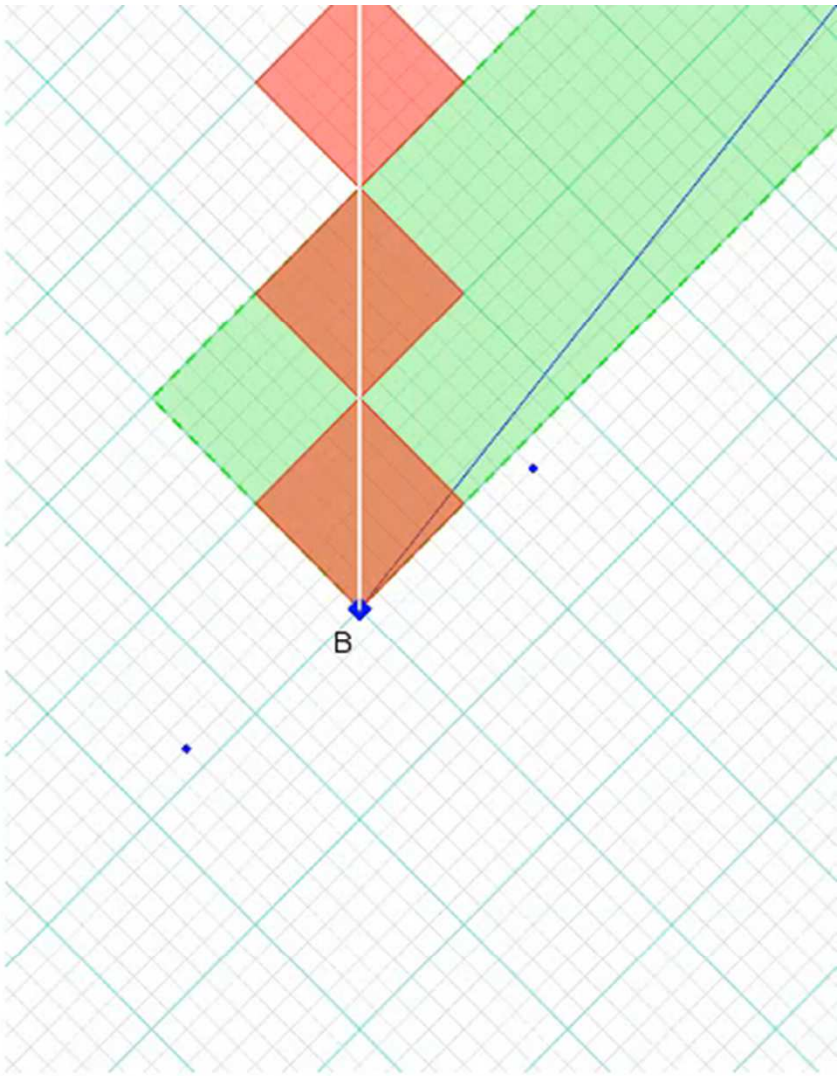




The causal diamond is an “18 by 2” grid using the lab’s clock diamonds,
but is a “6 by 6” using the muon’s clock diamonds ( as shown ).

*skippable*

# Again, Diamond diagonals define how "space ⊥ time"

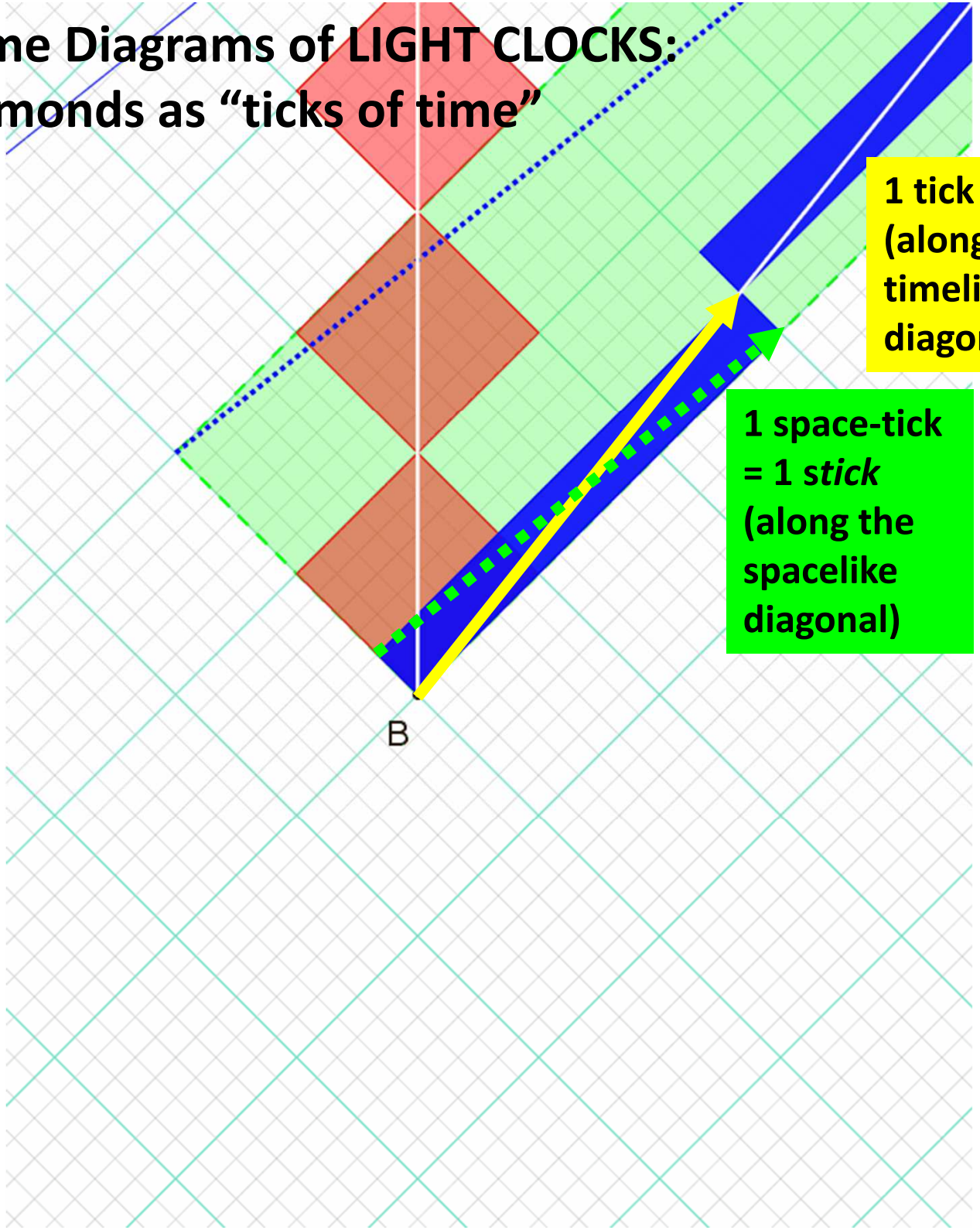


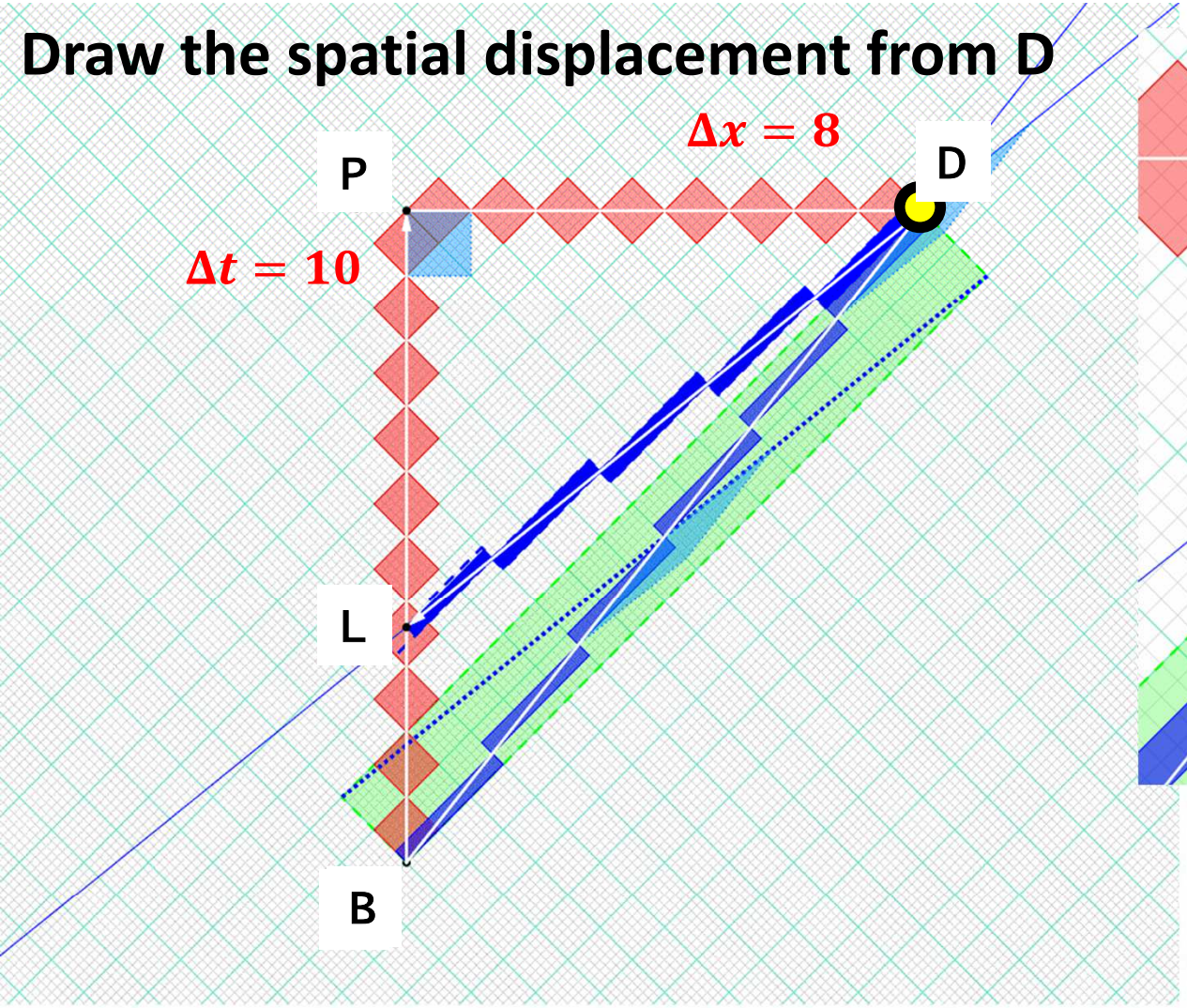

"A pair of events that are **simultaneous** for the inertial observer along BD"
lie on lines parallel to the
spacelike diagonal of the diamond BD.

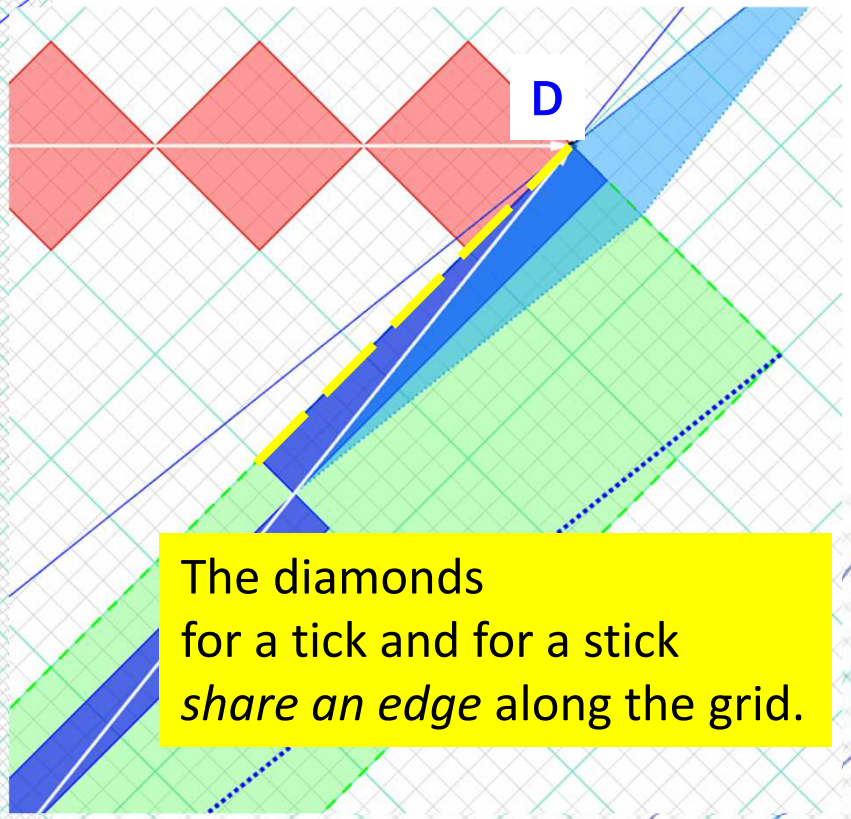
The spacelike diagonal (the spaceline)
is **perpendicular** to the timelike diagonal (the worldline [or, better, timeline]).
This captures the notion: "**space ⊥ time**".

# Again, Spacetime Diagrams of LIGHT CLOCKS: Light Clock Diamonds as "ticks of time"



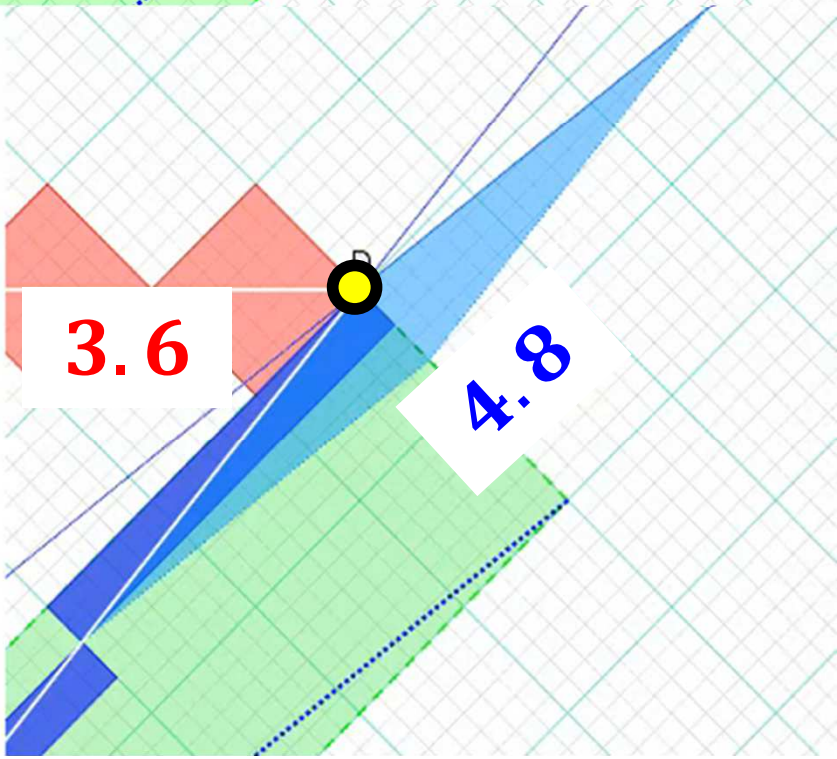
# Draw the spatial displacement from D



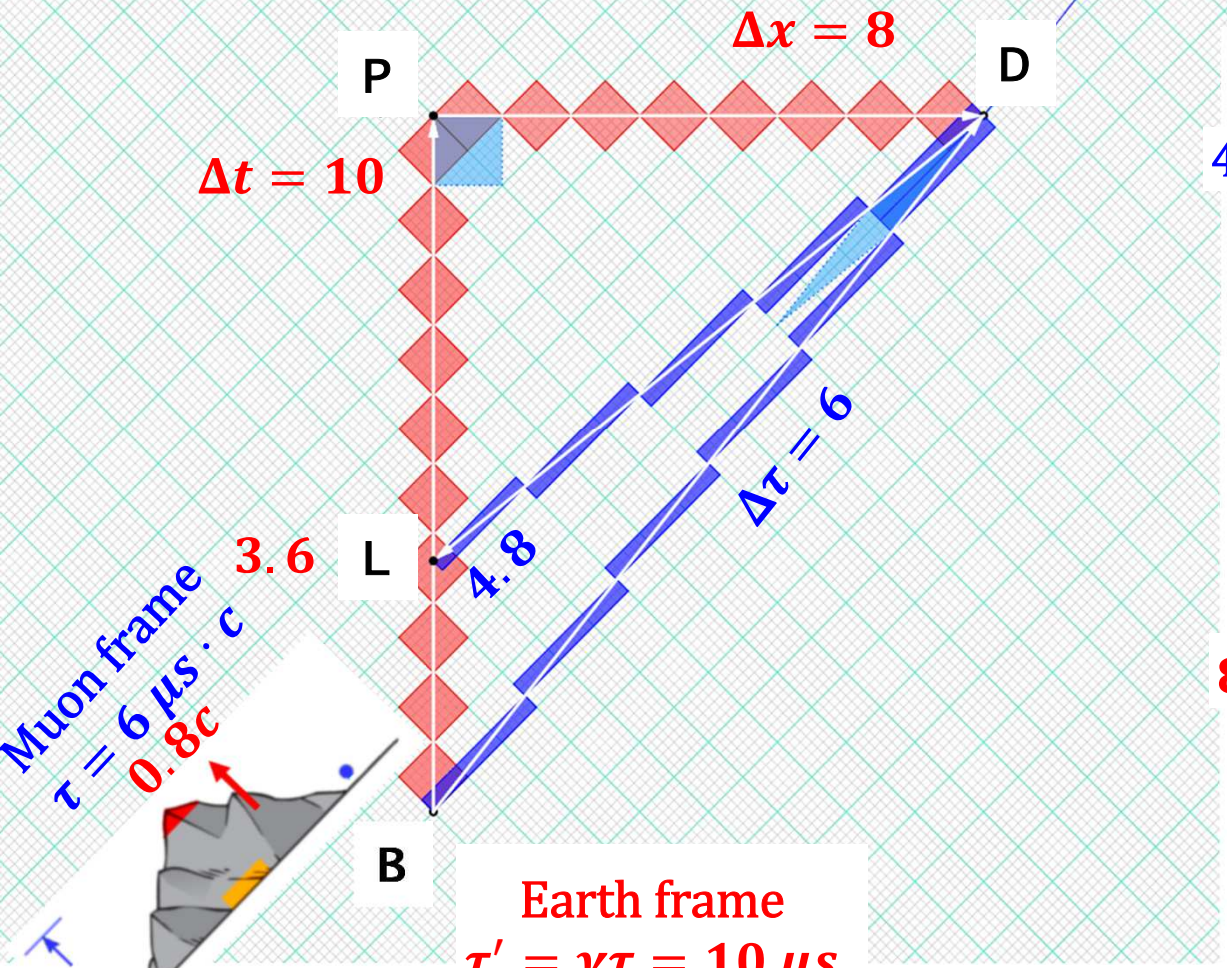



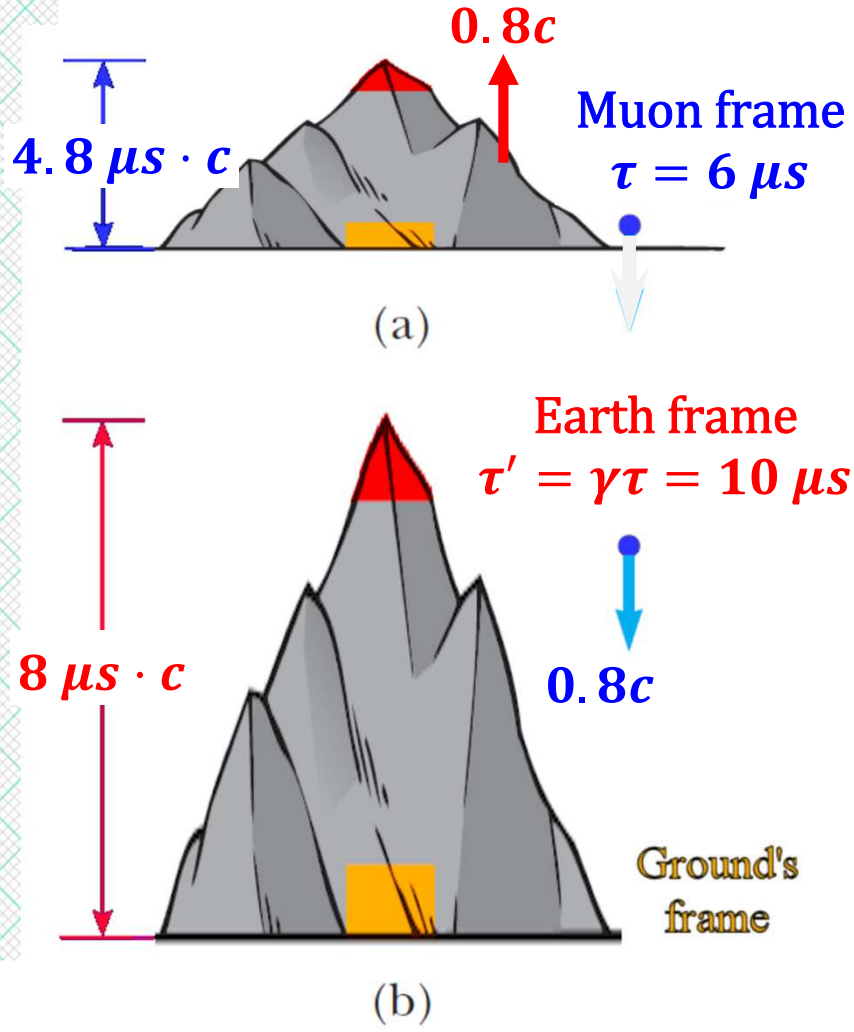

The diamonds
for a tick and for a stick
*share an edge* along the grid.

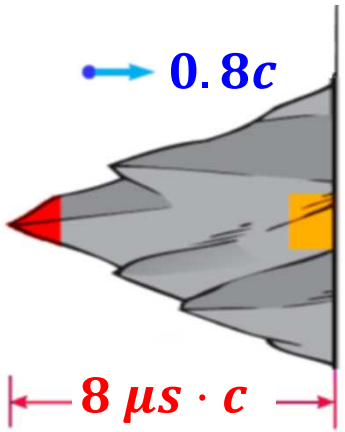
$DL \perp_{SR} BD$



# A cleaned-up spacetime diagram for the simplified muon problem



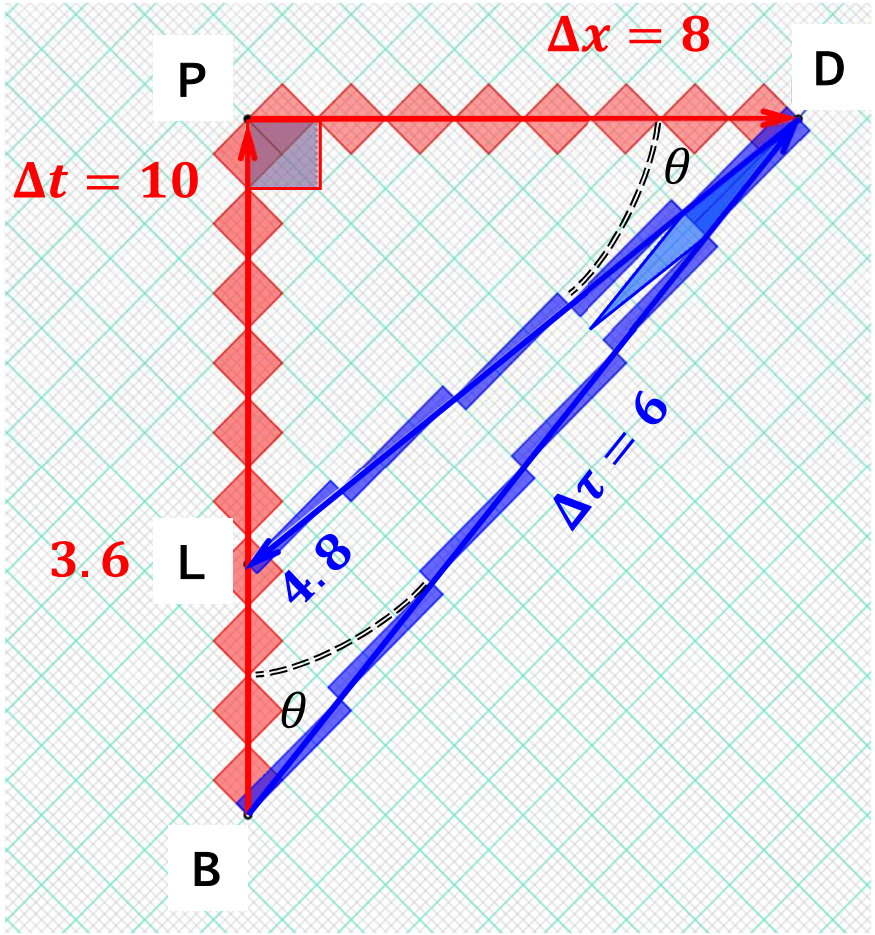

$$\frac{4.8\ \mu s \cdot c}{6\ \mu s} = 0.8c = \frac{8\ \mu s \cdot c}{10\ \mu s}$$

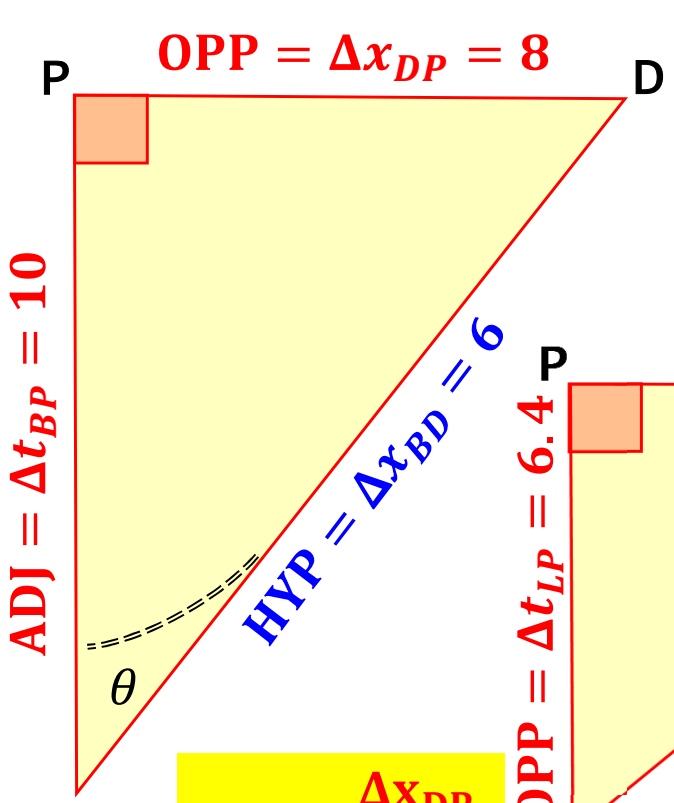
$$\frac{10\ \mu s}{6\ \mu s} = \gamma = \frac{8\ \mu s \cdot c}{4.8\ \mu s \cdot c}$$

$$\frac{OPP}{ADJ} = c\,TANH\,\theta = v$$

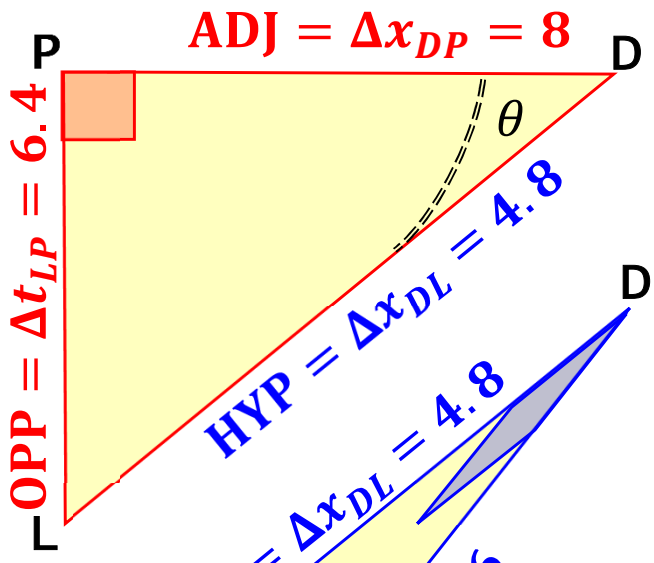
$$\frac{ADJ}{HYP} = COSH\,\theta = \gamma$$

# Some Hyperbolic-Trigonometry for the simplified muon problem




These are Minkowski-similar triangles



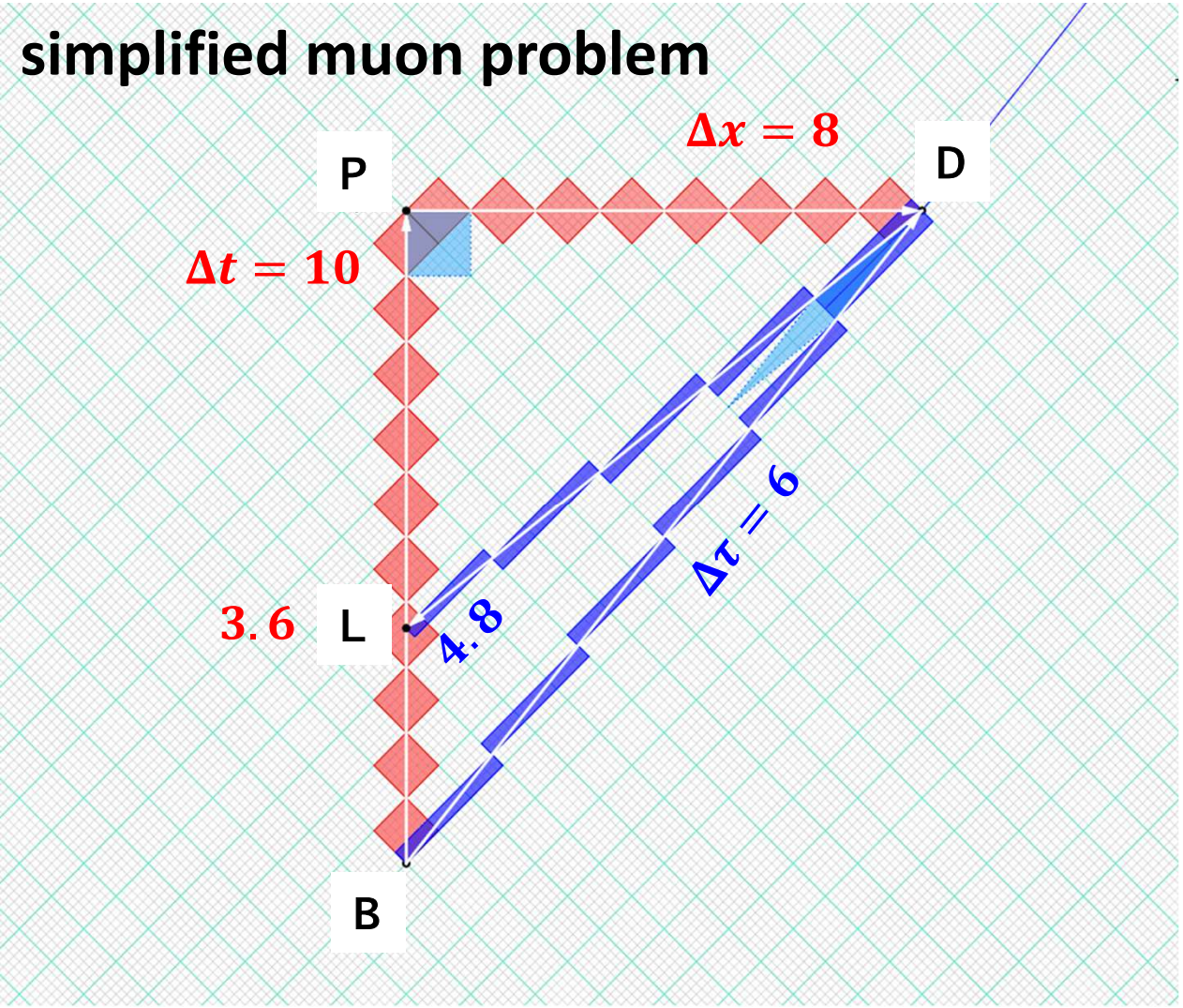

$$4.8 = \frac{\Delta x_{DP}}{\gamma}$$



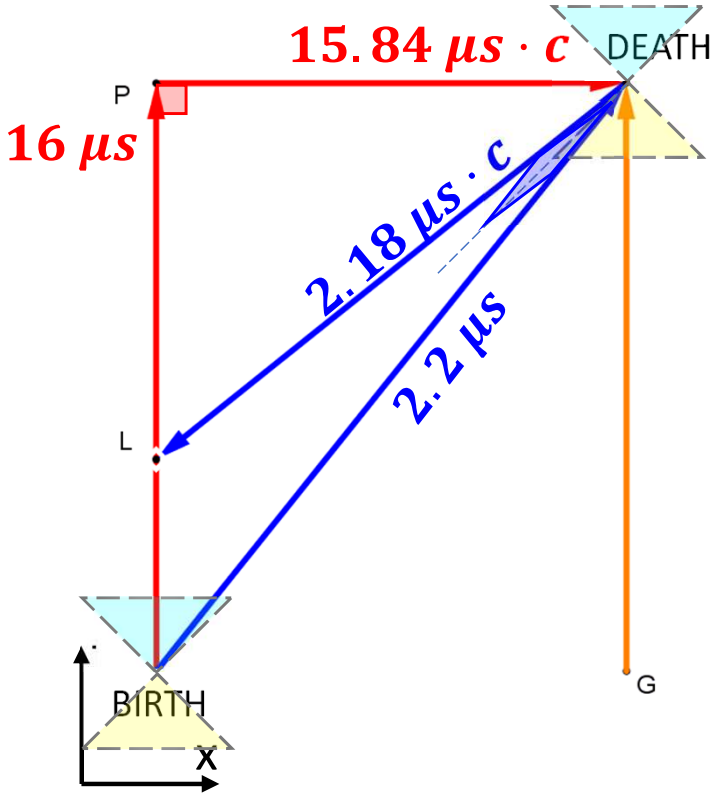

$$\gamma = COSH\,\theta = \frac{ADJ}{HYP} = \frac{BP}{BD} = \frac{10\ \mu s}{6\ \mu s} = \frac{DP}{DL} = \frac{8\ \mu s \cdot c}{4.8\ \mu s \cdot c} = \frac{BD}{BL} = \frac{6\ \mu s}{3.6\ \mu s}$$

$$3.6 = \frac{\Delta\tau_{BD}}{\gamma}$$

## simplified muon problem




## real muon problem




$$\frac{4.8\ \mu s \cdot c}{6\ \mu s} = 0.8c = \frac{8\ \mu s \cdot c}{10\ \mu s}$$

$$\frac{OPP}{ADJ} = c\ TANH\ \theta = v$$

$$\frac{2.18\ \mu s \cdot c}{2.2\ \mu s} = 0.99c = \frac{15.84\ \mu s \cdot c}{16\ \mu s}$$

$$\frac{10\ \mu s}{6\ \mu s} = \gamma = \frac{8\ \mu s \cdot c}{4.8\ \mu s \cdot c}$$

$$\frac{ADJ}{HYP} = COSH\ \theta = \gamma$$

$$\frac{16\ \mu s}{2.2\ \mu s} = \gamma = \frac{15.84\ \mu s \cdot c}{2.18\ \mu s \cdot c}$$

The **Muon Experiment** problem in Special Relativity was treated using "**Relativity on Rotated Graph Paper"**
American Journal of Physics **84**, 344 (2016);
https://doi.org/10.1119/1.4943251



- **By visualizing tickmarks** on a spacetime diagram, it is possible to **calculate graphically** by **counting ticks** and doing **simple algebra**.
- We hope to develop this further for an introductory ***algebra-based*** physics course.



***additional resources on the method:***

blog contribution: https://www.physicsforums.com/insights/relativity-rotated-graph-paper/
visualizations using GeoGebra: https://www.geogebra.org/m/HYD7hB9v#
early draft: https://arxiv.org/abs/1111.7254





# Summary for the simplified muon

$$\overset{\checkmark}{\Delta t_{BD}} = \overset{\checkmark}{\Delta\tau} = 6\ \mu s\ , \quad \overset{\checkmark}{v} = 0.8c = \left(\frac{4}{5}c\right),$$

$$\overset{?}{\gamma} = \frac{(ADJ)}{(HYP)} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = 5/3$$

$$\overset{?}{(ADJ)} \overset{\Delta BPD}{=} \gamma\,(HYP):$$

$$\overset{?}{\Delta t_{BP}} \overset{\text{sim}}{=} \Delta t_{BD} \overset{\text{td}}{=} \gamma\Delta\tau_{BD} = \left(\frac{5}{3}\right)(6\ \mu s) = 10\ \mu s$$

$$\overset{?}{(OPP)} \overset{\Delta BPD}{=} v\,(ADJ)$$

$$\overset{?}{\Delta x_{PD}} \overset{\text{vel}}{=} v\,\Delta t_{PD} = \left(\frac{4}{5}c\right)(10\ \mu s) = 8\ \mu s \cdot c$$

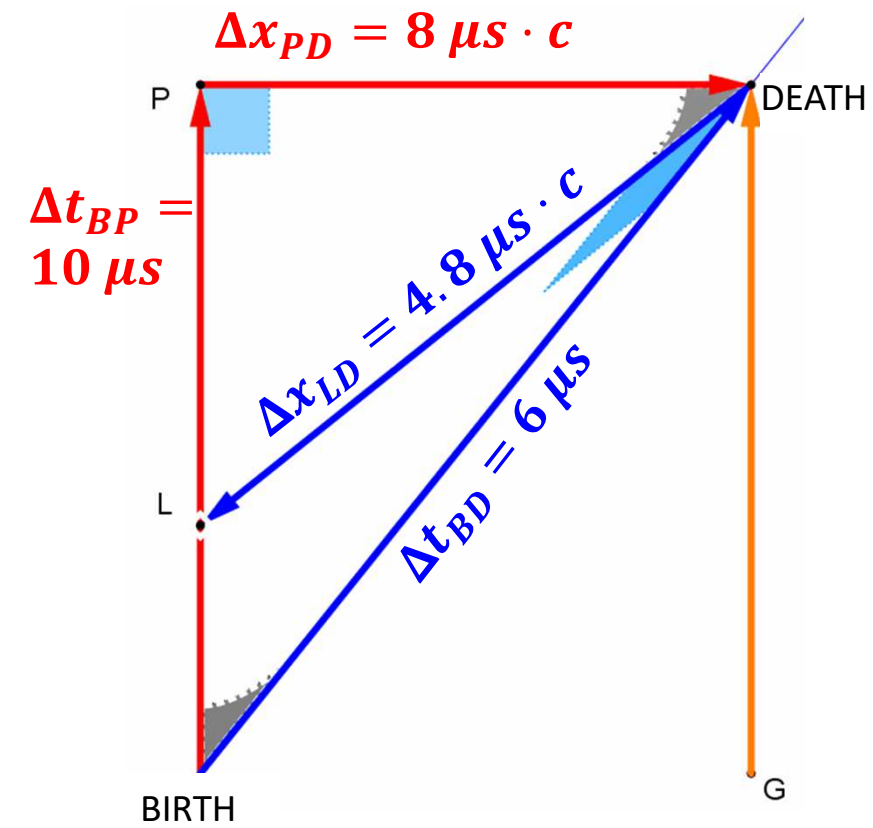
$$(ADJ) \overset{\Delta DPL}{=} \gamma(\overset{?}{HYP})$$

$$\Delta x_{PD} \overset{\text{lc}}{=} \gamma\,\overset{?}{\Delta x_{LD}}$$

$$\overset{?}{\Delta x_{LD}} \overset{\text{lc}}{=} \frac{\Delta x_{PD}}{\gamma} = \frac{8}{(5/3)} = \frac{24}{5}\mu s \cdot c = 4.8\ \mu s \cdot c$$

$$\overset{?}{(OPP)} \overset{\Delta BDL}{=} v\,(ADJ)$$

$$\overset{?}{\Delta x_{LD}} \overset{\text{vel}}{=} v\,\Delta\tau_{BD} = \left(\frac{4}{5}c\right)(6\ \mu s) = 4.8\ \mu s \cdot c$$



# Summary for the realistic muon

$\overset{\checkmark}{\Delta t_{BD}} = \overset{\checkmark}{\Delta\tau} = 2.2\,\mu s$ , $\overset{\checkmark}{v} = 0.99c$,

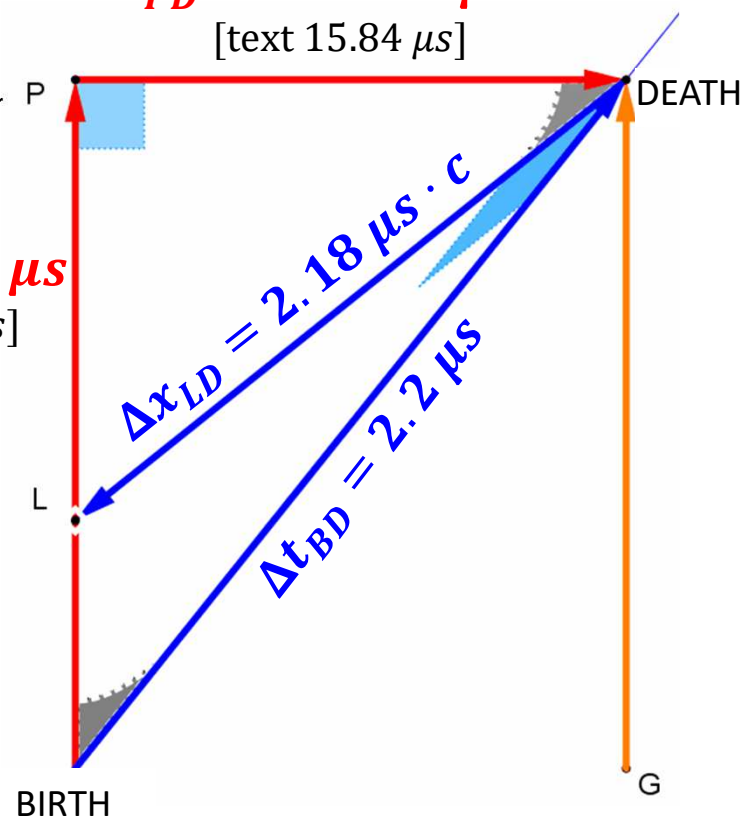
$\overset{?}{\gamma} = \frac{(ADJ)}{(HYP)} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = 7.08881$

$\overset{?}{(ADJ)} \overset{\Delta BPD}{=} \gamma\,(HYP)$:

$\overset{?}{\Delta t_{BP}} \overset{\text{sim}}{=} \Delta t_{BD} \overset{\text{td}}{=} \gamma\Delta\tau_{BD} = 15.59539\,\mu s$

$\Delta t_{BP} = 15.595\,\mu s$
[text 16 $\mu s$]

$\Delta x_{PD} = 15.439\,\mu s \cdot c$
[text 15.84 $\mu s$]




$\overset{?}{(OPP)} \overset{\Delta BPD}{=} v\,(ADJ)$

$\overset{?}{\Delta x_{PD}} \overset{\text{vel}}{=} v\,\Delta t_{PD} = 15.43943\,\mu s \cdot c = 4931.8\ \text{m}$

$(ADJ) \overset{\Delta DPL}{=} \gamma(\overset{?}{HYP})$

$\Delta x_{PD} \overset{\text{lc}}{=} \gamma\,\overset{?}{\Delta x_{LD}}$

$\overset{?}{\Delta x_{LD}} \overset{\text{lc}}{=} \frac{\Delta x_{PD}}{\gamma} = \left(\frac{15.43943}{7.08881}\right) = 2.178\,\mu s \cdot c = 653.4\ \text{m}$

$\overset{?}{(OPP)} \overset{\Delta BDL}{=} v\,(ADJ)$

$\overset{?}{\Delta x_{LD}} \overset{\text{vel}}{=} v\,\Delta\tau_{BD} = (0.99c)(2.2\,\mu s) = 2.178\,\mu s \cdot c$