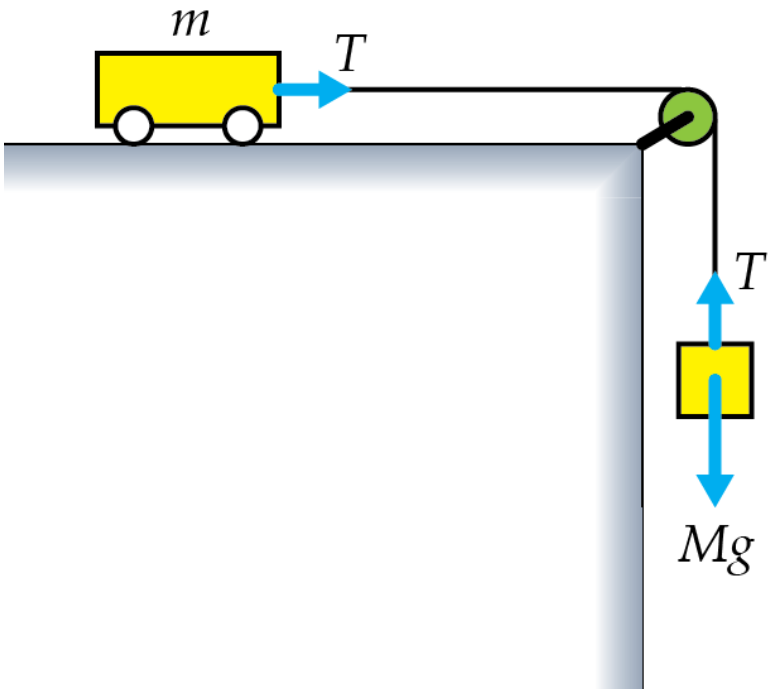
A Variation of the Cart, String, Pulley, and Weight Problem

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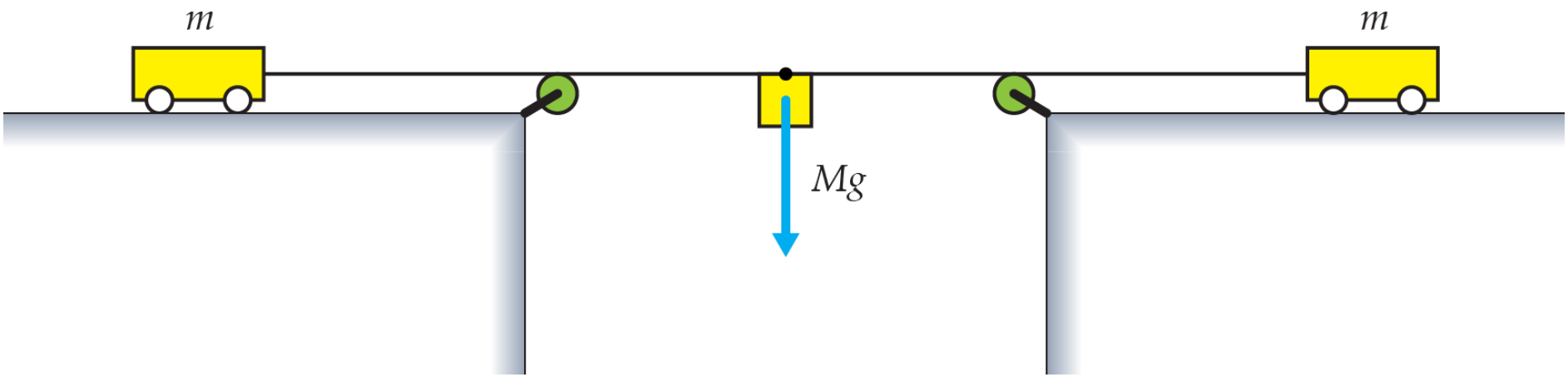
CSAAPT Fall 2026 Meeting W&M, October 10, 2026

The Standard Problem:
What is the acceleration of the cart and weight?



$$
\begin{aligned}
Ma &= Mg - T \\
ma &= T \\
&\downarrow \\
a &= \frac{Mg}{M+m}
\end{aligned}
$$

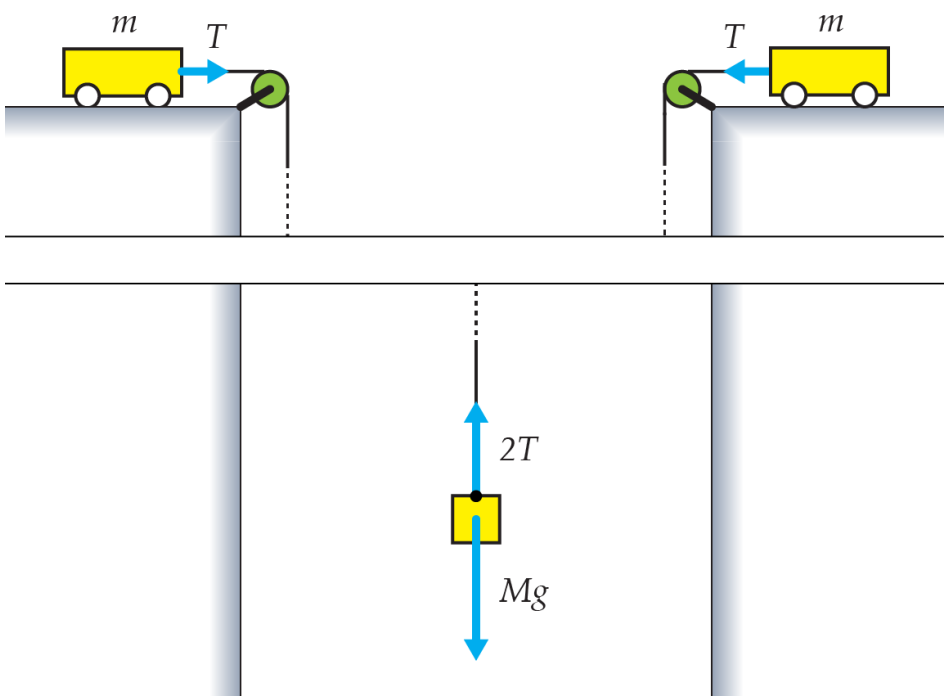
Variation, Question 1: What are the accelerations of the carts and weight at time $t = 0$?



$$a_{\text{weight}} = g$$
$$a_{\text{carts}} = 0$$

Variation, Question 2: What are the accelerations of the carts and weight as $t \to \infty$?

Assume that the string is so long that the carts never hit the pulleys.



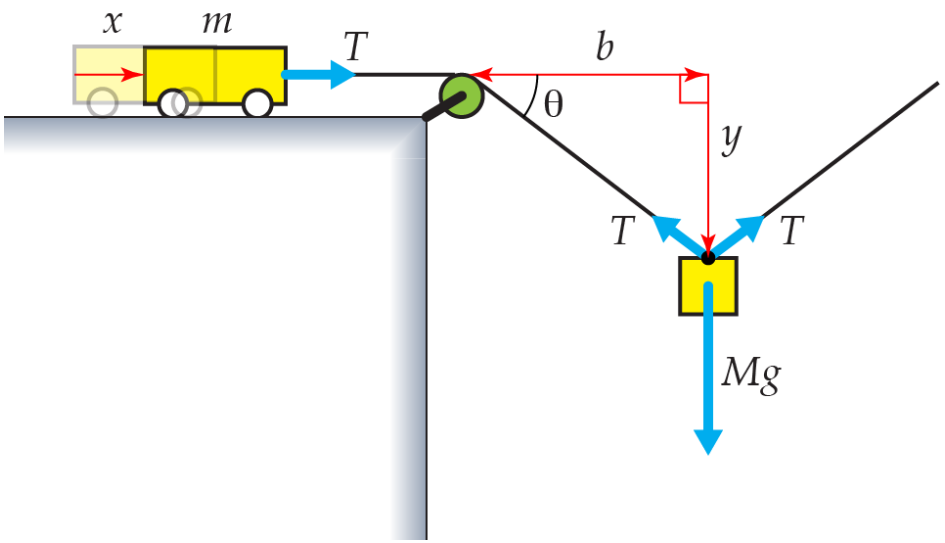
$$
\begin{aligned}
Ma &= Mg - 2T \\
ma &= T \\
&\downarrow \\
a &= \underbrace{\left(\frac{M}{M+2m}\right)}_{\lambda} g \\
&= \lambda g
\end{aligned}
$$

Variation, Question 3:

- Initially, only the weight will accelerate
- The weight will continue to accelerate and never slow down
- As $t \to \infty$, the weight and cart will be moving with the same velocity and same acceleration
- The acceleration of the carts must exceed the acceleration of the weight somewhere for them to catch up with the weight. But how? and where?

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Constraint from the unstretchable string:



Length of string is constant:

$$\begin{aligned} x &= \sqrt{y^2 + b^2} - b \\ &\downarrow \\ \dot{x} &= \frac{y\dot{y}}{\sqrt{y^2 + b^2}} = \dot{y}\sin\theta \end{aligned}$$

As θ evolves from $\theta = 0$ to $\theta \to \frac{\pi}{2}$, $\dot{x}$ will catch up to $\dot{y}$.

Energy Conservation:

Energy conservation:

$$\begin{aligned} Mgy &= \frac{1}{2}M\dot{y}^2 + m\dot{x}^2 \\ &= \left(\frac{M}{2} + \frac{my^2}{y^2+b^2}\right)\dot{y}^2 \\ &\downarrow \\ \dot{y}^2 &= \frac{2\lambda g\, y(y^2+b^2)}{y^2+\lambda b^2} \\ \dot{x}^2 &= \frac{2\lambda g\, y^3}{y^2+\lambda b^2} \end{aligned}$$

where

$$\lambda = \frac{M}{M+2m}$$

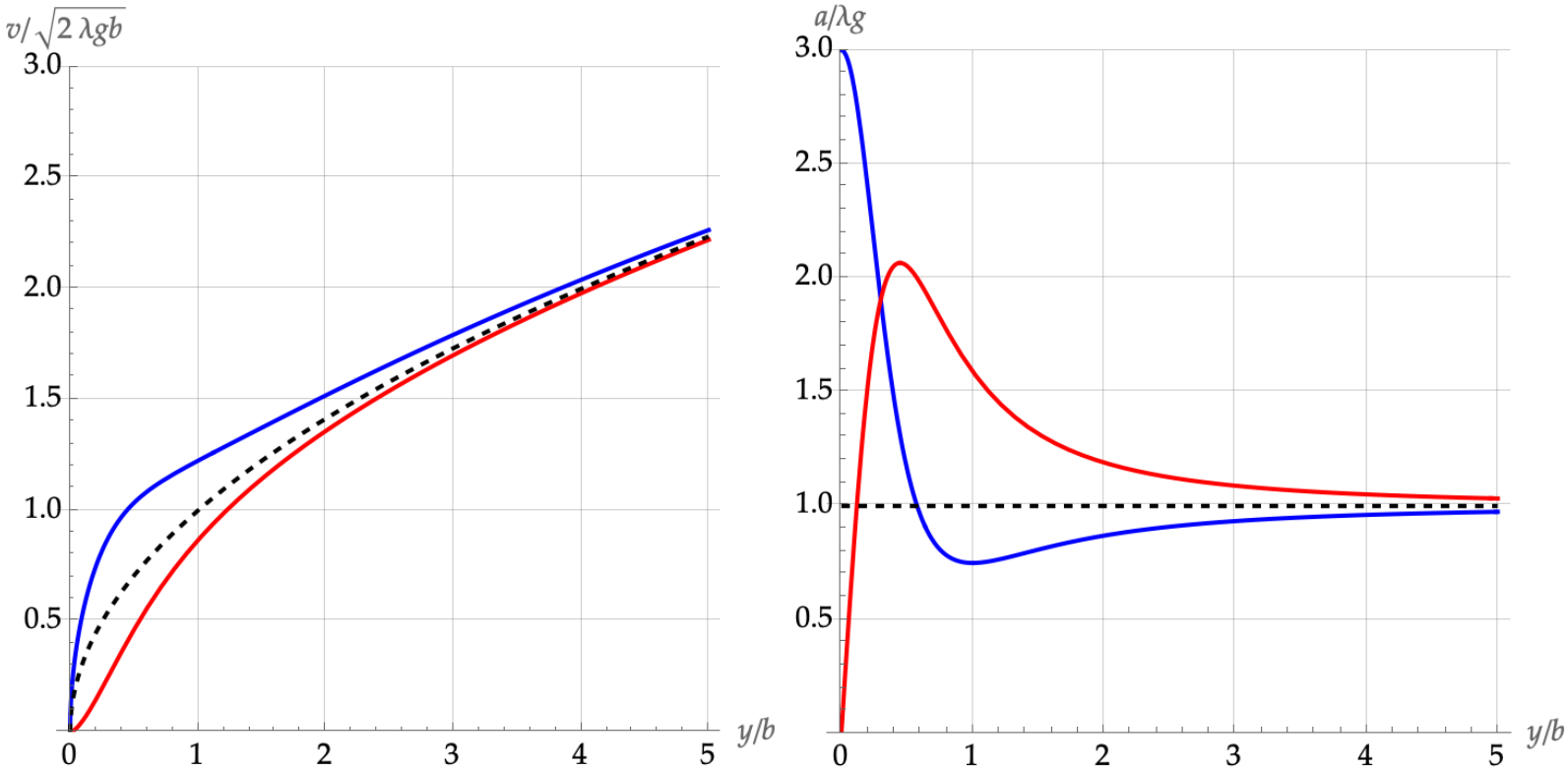
Differentiate with time to derive the accelerations:

$$\begin{aligned} \frac{\ddot{y}}{\lambda g} &= \frac{y^4 + (3\lambda-1)b^2y^2 + \lambda b^4}{(y^2+\lambda b^2)^2} \\ \frac{\ddot{x}}{\lambda g} &= \frac{y(y^2+3\lambda b^2)\sqrt{y^2+b^2}}{(y^2+\lambda b)^2} \end{aligned}$$

Note that as $y \to \infty$ or $b \to 0$, both $\ddot{x}$ and $\ddot{y}$ converge to λg as they should. But what do these functions look like for finite y?

Example: $M = m$ case

Consider the $M = m$ case for which $\lambda = \frac{1}{3}$. The graphs of the velocities and accelerations as functions of y look like:



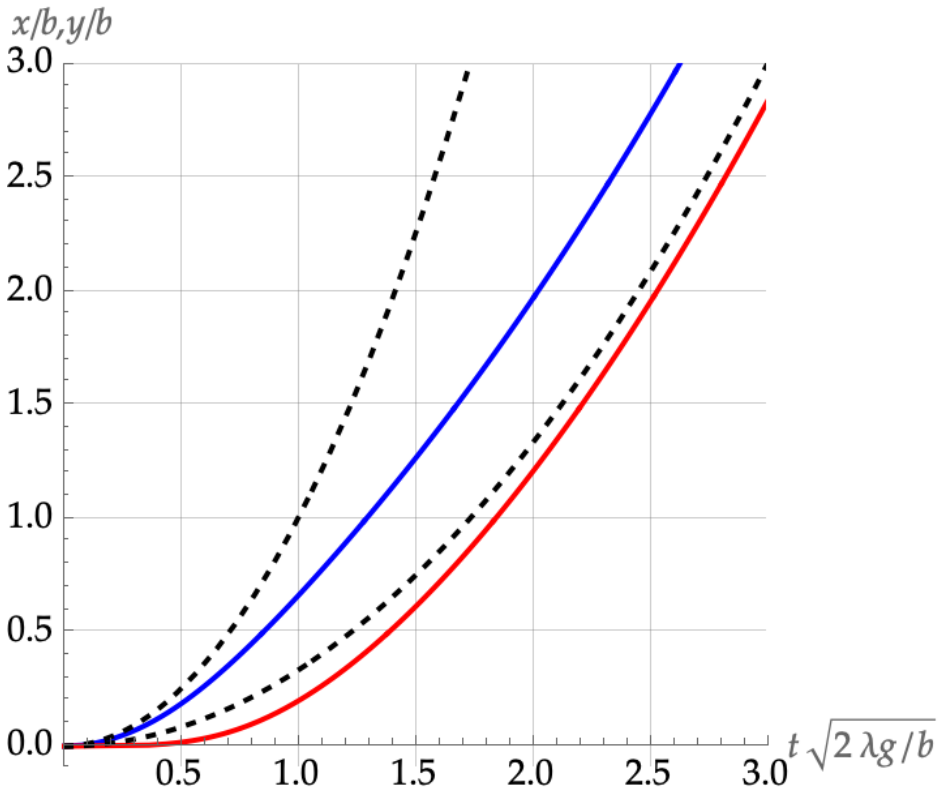
The blue lines are $\dot{y}$ and $\ddot{y}$, while the red lines are $\dot{x}$ and $\ddot{x}$.

Time Dependence:

$$\begin{aligned}
\frac{\dot{y}}{\sqrt{2\lambda g b}} &= \sqrt{\frac{y(y^2+b^2)}{b(y^2+\lambda b^2)}} \\
&\downarrow \\
\int dt &= \sqrt{\frac{b}{2\lambda g}} \int \frac{dy}{b} \sqrt{\frac{b(y^2+\lambda b^2)}{y(y^2+b^2)}}
\end{aligned}$$

$$\begin{aligned}
\frac{\dot{x}}{\sqrt{2\lambda g b}} &= \sqrt{\frac{y^3}{b(y^2+\lambda b^2)}} = \sqrt{\frac{(x^2+2bx)^{3/2}}{x^2+2bx+\lambda b^2}} \\
&\downarrow \\
\int dt &= \sqrt{\frac{b}{2\lambda g}} \int \frac{dx}{b} \sqrt{\frac{x^2+2bx+\lambda b^2}{(x^2+2bx)^{3/2}}}
\end{aligned}$$

Time Dependence:



The weight (blue) takes off first and the carts (red) follow behind.

The End