QUANTUM MECHANICS

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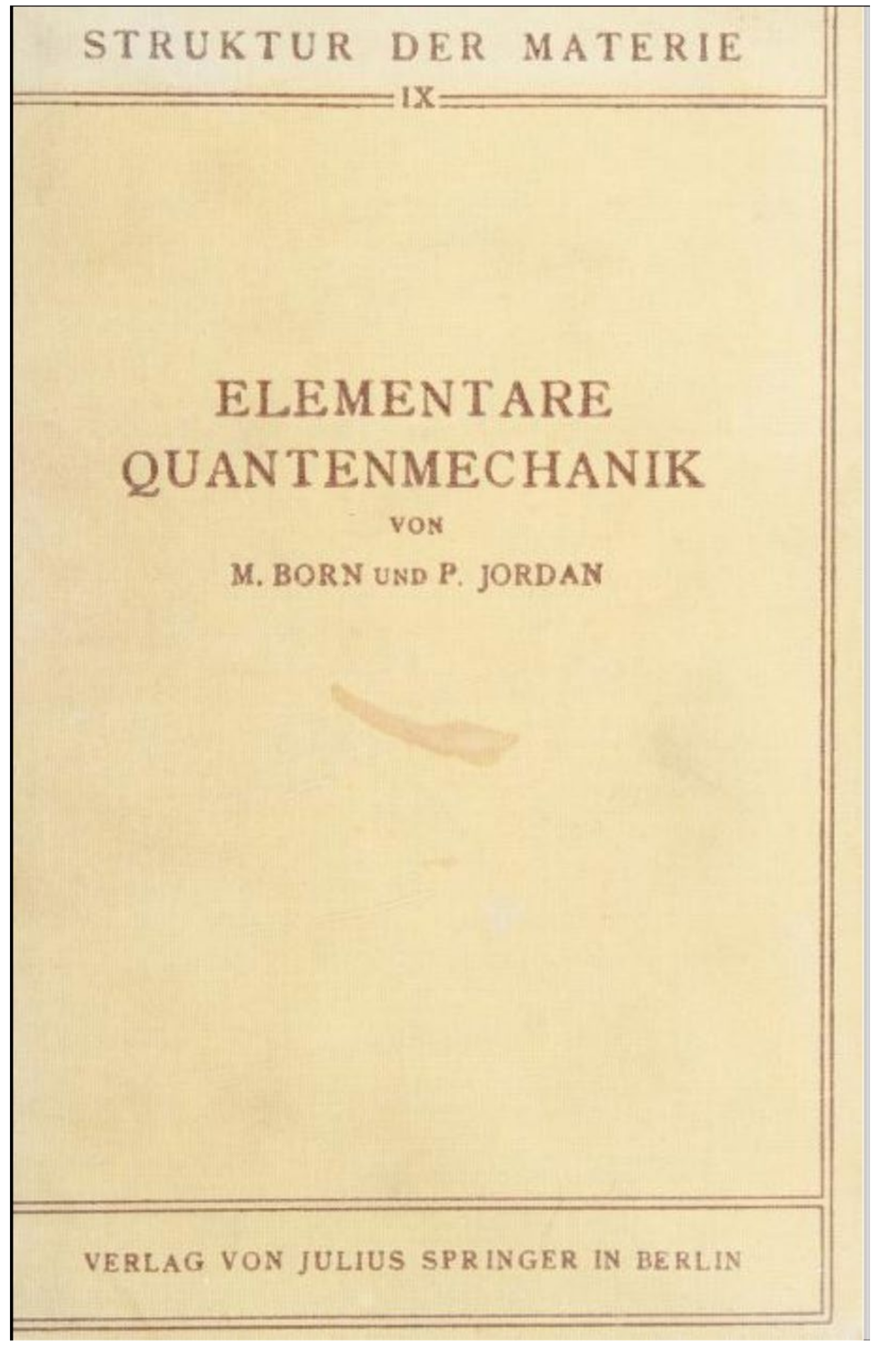
Complex numbers are often introduced in mechanics as a tool for solving driven-damped harmonic oscillators

Usually this is presented as a set of rules with an uneasy "take the real part of the solution" at the end

In this talk, I will show you a more natural way to organize working with complex numbers and driven-damped harmonic oscillators

The idea for this comes from Born and Jordan's Elementare Quantenmechanik (1930)



Die kanonischen Bewegungsgleichungen

(3) $$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{\mu}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -a q$$

oder, mit der Abkürzung (2):

(3a) $$\ddot{q} = -(2\pi\nu_0)^2 q,$$

lauten wie in der klassischen Theorie. Definiert man „komplexe Amplituden“

(4) $$\begin{cases} b = C(p - 2\pi i \nu_0 \mu q), \\ b^\dagger = C(p + 2\pi i \nu_0 \mu q) \end{cases}$$

mit einer vorläufig beliebigen Konstanten C, so gehen die Gleichungen (3) über in

(5) $$\dot{b} = -2\pi i \nu_0 b, \qquad \dot{b}^\dagger = 2\pi i \nu_0 b^\dagger.$$

Conventional approach (from Feynman)

square root. The only bothersome thing is that we get *two* solutions! Thus

$$\alpha_1 = i\gamma/2 + \sqrt{\omega_0^2 - \gamma^2/4} = i\gamma/2 + \omega_\gamma \tag{24.14}$$

and

$$\alpha_2 = i\gamma/2 - \sqrt{\omega_0^2 - \gamma^2/4} = i\gamma/2 - \omega_\gamma. \tag{24.15}$$

Let us consider the first one, supposing that we had not noticed that the square root has two possible values. Then we know that a solution for x is $x_1 = Ae^{i\alpha_1 t}$, where A is any constant whatever. Now, in substituting α_1, because it is going to come so many times and it takes so long to write, we shall call $\sqrt{\omega_0^2 - \gamma^2/4} = \omega_\gamma$. Thus $i\alpha_1 = -\gamma/2 + i\omega_\gamma$, and we get $x = Ae^{(-\gamma/2 + i\omega_\gamma)t}$, or what is the same, because of the wonderful properties of an exponential,

$$x_1 = Ae^{-\gamma t/2}e^{i\omega_\gamma t}. \tag{24.16}$$

First, we recognize this as an oscillation, an oscillation at a frequency ω_γ, which is not *exactly* the frequency ω_0, but is rather close to ω_0 if it is a good system. Second, the amplitude of the oscillation is decreasing exponentially! If we take, for instance, the real part of (24.16), we get

$$x_1 = Ae^{-\gamma t/2}\cos\omega_\gamma t. \tag{24.17}$$

This is very much like our guessed-at solution (24.10), except that the frequency really is ω_γ. This is the only error, so it is the same thing—we have the right idea. But everything is *not* all right! What is not all right is that *there is another solution*.

The other solution is α_2, and we see that the difference is only that the sign of ω_γ is reversed:

$$x_2 = Be^{-\gamma t/2}e^{-i\omega_\gamma t}. \tag{24.18}$$

What does this mean? We shall soon prove that if x_1 and x_2 are each a possible solution of Eq. (24.1) with $F = 0$, then $x_1 + x_2$ is also a solution of the same equation! So the general solution x is of the mathematical form

$$x = e^{-\gamma t/2}(Ae^{i\omega_\gamma t} + Be^{-i\omega_\gamma t}). \tag{24.19}$$

Simple Harmonic Oscillator

Hamilton equation of motion: $\dot{x} = \frac{p}{m}$ and $\dot{p} = F = -m\omega^2 x$

Decouple the differential equation: Let $A = \alpha x + p$ and find α such that $\dot{A} = cA$

Decouple the differential equation: $\dot{A} = \alpha\dot{x} + \dot{p} = \alpha\frac{p}{m} - m\omega^2 x$ implies $c = \frac{\alpha}{m}$ and $c\alpha = -m\omega^2$

Solve for α and c: $-m\omega^2 = \frac{\alpha^2}{m}$ or $\alpha^2 = -m^2\omega^2$ or $\alpha = \pm im\omega$ and $c = \pm i\omega$

So we have $A = p - im\omega x$ and $\dot{A} = -i\omega A$ or $A = A_0 e^{-i\omega t}$ with $A_0 = p_0 - im\omega x_0$

The other solution is $A^* = p + im\omega x$ and $\dot{A}^* = i\omega A$ or $A^* = A_0^* e^{i\omega t}$ with $A_0^* = p_0 + im\omega x_0$

Solve for x: $x = \frac{-A+A^*}{2im\omega} = x_0 \cos\omega t + \frac{p_0}{m\omega}\sin\omega t$

Complex numbers are introduced naturally in the process of finding a first-order differential equation.

Energy

We found $A = A_0 e^{-i\omega t}$ with $A_0 = p_0 - im\omega x_0$ and $A^* = A_0^* e^{i\omega t}$ with $A_0^* = p_0 + im\omega x_0$

So, we have $\frac{p^2(t)}{2m} + \frac{1}{2}m\omega^2 x^2(t) = \frac{1}{2m}A^*(t)A(t) = \frac{1}{2m}|A_0|^2 = \frac{p_0^2}{2m} + \frac{1}{2}m\omega^2 x_0^2 = \text{constant}$

So, energy is conserved, automatically in this approach

Relation to quantum mechanics

We found $A = p - im\omega x$ and $A^* = p + im\omega$

If we make them operators, with $[\hat{x}, \hat{p}] = i\hbar$, then $\frac{1}{2m}\hat{A}^\dagger \hat{A} + \frac{1}{2}\hbar\omega = \widehat{H}$

Hence, this approach makes an easy transition to raising and lowering operators in quantum mechanics.

Damped Harmonic Oscillator

Hamilton equation of motion: $\dot{x} = \frac{p}{m}$ and $\dot{p} = F = -m\gamma\dot{x} - m\omega^2 x$

Decouple the differential equation: Let $A = \alpha x + p$ and find α such that $\dot{A} = cA$

Decouple the differential equation: $\dot{A} = \alpha\dot{x} + \dot{p} = \alpha\frac{p}{m} - \gamma p - m\omega^2 x$ implies $c = \frac{\alpha}{m} - \gamma$ and $c\alpha = -m\omega^2$

Solve for α and c: $-m\omega^2 = \frac{\alpha^2}{m} - \gamma\alpha$ or $\alpha^2 - \gamma\alpha + m^2\omega^2 = 0$ or $\alpha = \frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 4m^2\omega^2}$ and

$c = -\frac{m\omega^2}{\alpha}$

So we have $A = p - im\omega\sqrt{1 - \frac{\gamma^2}{4m^2\omega^2}}\,x$ and $\dot{A} = \left(-\frac{\gamma}{2m} - i\omega\sqrt{1 - \frac{\gamma^2}{4m^2\omega^2}}\right)A$

This gives the standard damped solution. You can look at energy as well, but the analysis is more complicated.

Driven Damped Harmonic Oscillator

Hamilton equation of motion: $\dot{x} = \frac{p}{m}$ and $\dot{p} = F = -m\gamma\dot{x} - m\omega^2 x + F(t)$

Now we can derive an inhomogeneous first-order differential equation. It is easy to show students how to solve this via an integrating factor. But there is not enough time to go into the details in full.

For me, this is a much more natural way to introduce complex numbers and the added connection to quantum mechanics is another bonus

References

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These ideas can be extended to Kepler orbits and be linked to the quantum solutions of hydrogen. But that is for the Fall meeting.

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