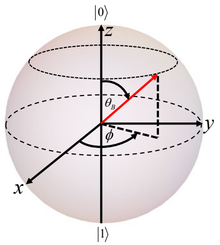


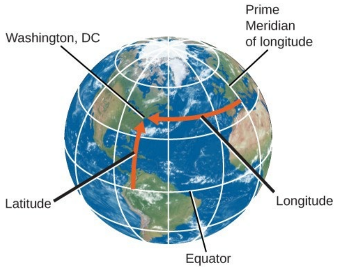


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# Improved Pedagogical Introduction to Quantum Entangled States


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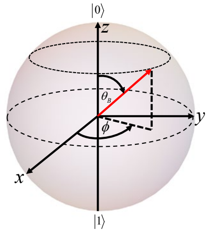


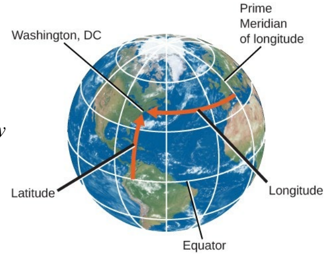


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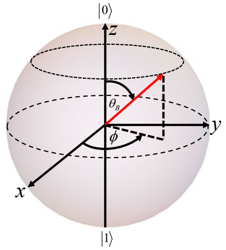
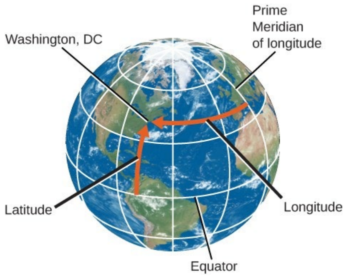
# Introduction and Overview

- Entangled States – what's the big deal?
- Need pedagogy that diminishes student misconceptions and misunderstandings
- Prepares for QIS and QC Applications



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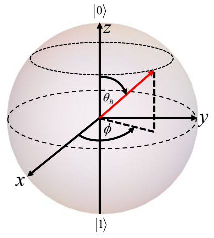
# Multi-particle Quantum States

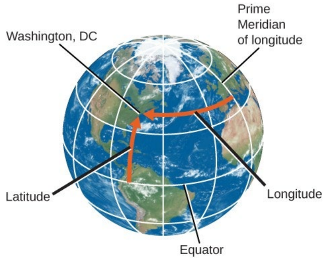
## Example: two qubits

- Two spin-1/2 systems (not necessarily of the same type of particle)
- Four-dimensional Hilbert space = # of basis states
- Use basis states for spin along z axis.
- For the composite system

$$\left|\uparrow_A\right\rangle \otimes \left|\uparrow_B\right\rangle = \left|\uparrow_A \uparrow_B\right\rangle$$

$$\left|\psi\right\rangle = c_{\uparrow\uparrow}\left|\uparrow_A \uparrow_B\right\rangle + c_{\uparrow\uparrow}\left|\uparrow_A \downarrow_B\right\rangle + c_{\uparrow\uparrow}\left|\downarrow_A \uparrow_B\right\rangle + c_{\uparrow\uparrow}\left|\downarrow_A \downarrow_B\right\rangle$$



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# Most general superposition state of the two-qubit composite system

$$|\psi\rangle = c_{\uparrow\uparrow}|\uparrow_A \uparrow_B\rangle + c_{\uparrow\downarrow}|\uparrow_A \downarrow_B\rangle + c_{\downarrow\uparrow}|\downarrow_A \uparrow_B\rangle + c_{\downarrow\downarrow}|\downarrow_A \downarrow_B\rangle$$

State coefficient

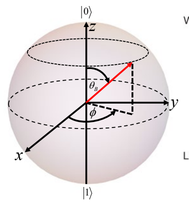
## Product State – special case

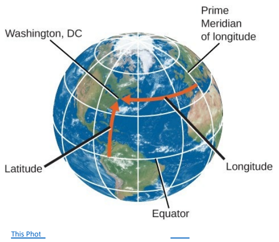
## Independent qubits

$$|\psi_{\text{prod}}\rangle = \alpha_A |\Psi_A\rangle \alpha_B |\Psi_B\rangle$$

$|\psi\rangle$ can be written as a product state if and only if $c_{\uparrow\uparrow} c_{\downarrow\downarrow} = c_{\uparrow\downarrow} c_{\downarrow\uparrow}$

- Otherwise, the composite state is said to be entangled.
- There are no quantum states describing the individual qubits.
- Correlations among measurements on the individual qubits.

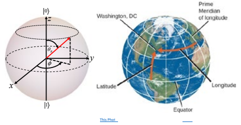



# Traditional Approach to Entanglement

- Tensor product space
- Need to define operators in that tensor space
- Operate on both qubits
- Operate on only one qubit, leave the other alone
- Need to specify the state of the system after measurement on only one part
- Need correlation functions (expectation values of various operators) to extract entanglement features

Steep learning curve
Too many new ideas




# Better Way: Focus on Probabilities

Entanglement correlation information

$$P(\uparrow_A \cap \uparrow_B) \qquad P(\uparrow_A) \qquad P(\uparrow_B | \uparrow_A)$$

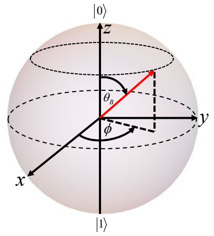
joint total conditional

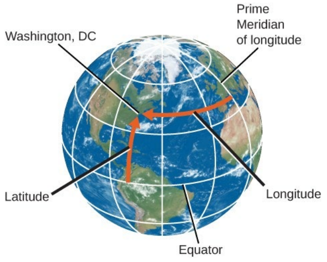
$P(\uparrow_A \cap \uparrow_B)$ Can be read off from composite system state vector

$$P(\uparrow_A) = P(\uparrow_A \cap \uparrow_B) + P(\uparrow_A \cap \downarrow_B)$$

$$P(\uparrow_A \cap \uparrow_B) = P(\uparrow_A | \uparrow_B) P(\uparrow_B) = P(\uparrow_B | \uparrow_A) P(\uparrow_A)$$

Bayes's Rule (Theorem)




$$\left|\psi\right\rangle = c_{\uparrow\uparrow}\left|\uparrow_A\uparrow_B\right\rangle + c_{\uparrow\downarrow}\left|\uparrow_A\downarrow_B\right\rangle + c_{\downarrow\uparrow}\left|\downarrow_A\uparrow_B\right\rangle + c_{\downarrow\downarrow}\left|\downarrow_A\downarrow_B\right\rangle$$

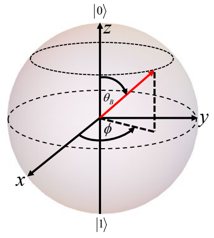
$$P(\uparrow_A \cap \uparrow_B) = \left|c_{\uparrow\uparrow}\right|^2$$

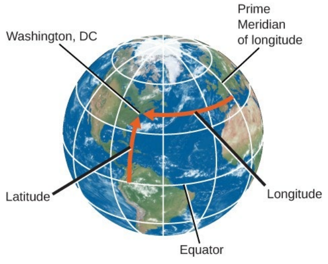
$$P(\uparrow_A) = \left|c_{\uparrow\uparrow}\right|^2 + \left|c_{\uparrow\downarrow}\right|^2$$

$$P(\uparrow_B | \uparrow_A) = \frac{P(\uparrow_A \cap \uparrow_B)}{P(\uparrow_A)} = \frac{\left|c_{\uparrow\uparrow}\right|^2}{\left|c_{\uparrow\uparrow}\right|^2 + \left|c_{\uparrow\downarrow}\right|^2}$$

$$P(\uparrow_B | \downarrow_A) = \frac{P(\downarrow_A \cap \uparrow_B)}{P(\downarrow_A)} = \frac{\left|c_{\downarrow\uparrow}\right|^2}{\left|c_{\downarrow\uparrow}\right|^2 + \left|c_{\downarrow\downarrow}\right|^2}$$

$\left|\psi\right\rangle$ can be written as a product state if and only $P\left(\uparrow_B | \uparrow_A\right) = P\left(\uparrow_B | \downarrow_A\right)$






$$|\psi\rangle = c_{\uparrow\uparrow}|\uparrow_A\uparrow_B\rangle + c_{\uparrow\downarrow}|\uparrow_A\downarrow_B\rangle + c_{\downarrow\uparrow}|\downarrow_A\uparrow_B\rangle + c_{\downarrow\downarrow}|\downarrow_A\downarrow_B\rangle$$

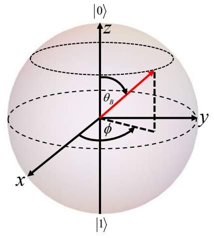
$$|\psi\rangle \text{ is an entangled state if and only } P\left(\uparrow_B|\uparrow_A\right) \neq P\left(\uparrow_B|\downarrow_A\right)$$

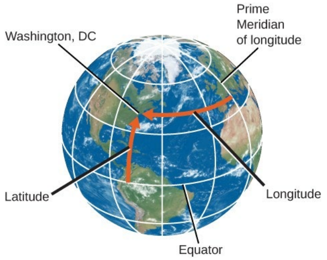
## Three equivalent statements:

The two qubits are not independent.
There are correlations among the measurements of A and B.
The composite state is entangled.

“Alice’s observations affect the probability of Bob’s measurement outcomes.”

“Spooky action at a distance”?






# Example

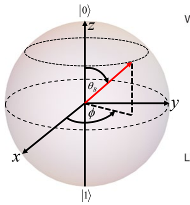
$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow_A \uparrow_B\rangle + |\uparrow_A \downarrow_B\rangle + |\downarrow_A \downarrow_B\rangle\right)$$

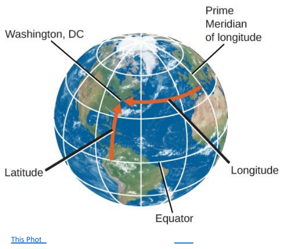
Is this an entangled state?

$$P\left(\uparrow_A \cap \uparrow_B\right) = P\left(\uparrow_A \cap \downarrow_B\right) = P\left(\downarrow_A \cap \downarrow_B\right) = \frac{1}{3} \qquad P\left(\downarrow_A \cap \uparrow_B\right) = 0$$

$$P\left(\uparrow_A\right) = \frac{2}{3} \qquad P\left(\downarrow_A\right) = \frac{1}{3} \qquad P\left(\uparrow_B\right) = \frac{1}{3} \qquad P\left(\downarrow_B\right) = \frac{2}{3}$$

$$P\left(\uparrow_B \middle| \uparrow_A\right) = \frac{1}{2} \qquad P\left(\uparrow_B \middle| \downarrow_A\right) = 0$$




# Summary

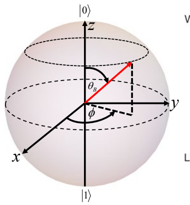
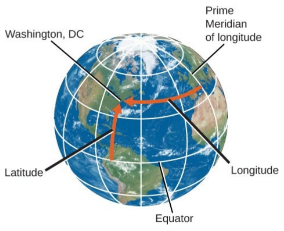
1. The composite system state vector contains all the information about measurement outcomes for the system: both collective and individual.

2. No need to invoke "collapse of the state vector."

3. Once the basics are in place, students are ready to learn about tensor products of operators, states after measurements, correlation functions, etc.

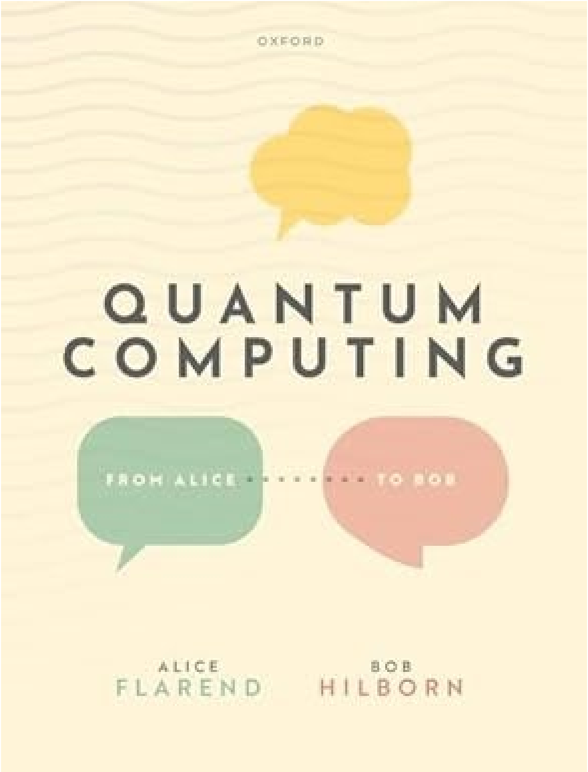
Applications:

Quantum State Teleportation
Bell's Theorem: (hidden variables vs quantum mechanics)
Entanglement Swapping
Superdense Coding


- Book: *Quantum Mechanics for Quantum Computing: A Linear Algebra Approach (fall 2025)*

Based on *Quantum Computing: From Alice to Bob,* Alice Flarend and Bob Hilborn (Oxford University Press, 2022)

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