Bernoulli Equation for Gases

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of the American Association of Physics Teachers

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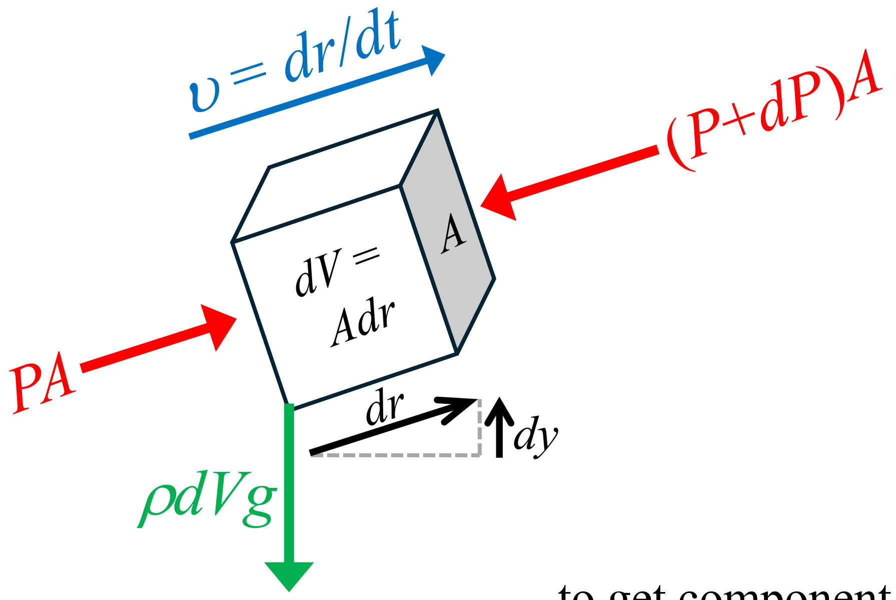
Learning Objectives

1. To show you a first-principles mathematical modification of the Bernoulli equation to treat compressible fluids.

2. To discuss that theory in light of two previous experiments about gases escaping out of a small hole in a pressurized vessel.

Push a block of fluid of volume dV at speed υ through a distance dr in a time dt:



$$F_{\text{net}} = ma \quad \Rightarrow \quad PA - (P + dP)A - \rho dVg \overbrace{\frac{dy}{dr}}^{\substack{\text{to get component of } mg \\ \text{in direction of motion}}} = \rho dVa$$

$$\div A \quad \Rightarrow \quad -dP - \rho g dy = \rho dr \frac{d\upsilon}{dt} \quad \Rightarrow \quad dP + \rho \upsilon d\upsilon + \rho g dy = 0$$

$$\div \rho \quad \Rightarrow \quad \frac{dP}{\rho} + \upsilon d\upsilon + g dy = 0 \quad -(1)$$

$$\frac{dP}{\rho} + \upsilon d\upsilon + g dy = 0 \quad -(1)$$

for an ideal gas $PV = nRT = \frac{m}{M} RT \quad -(2)$

multiply by $\frac{M}{RTV} \Rightarrow \rho = \frac{M}{RT} P \quad -(3)$

sub into (1) $\Rightarrow \frac{RT}{M}\frac{dP}{P} + \upsilon d\upsilon + g dy = 0$

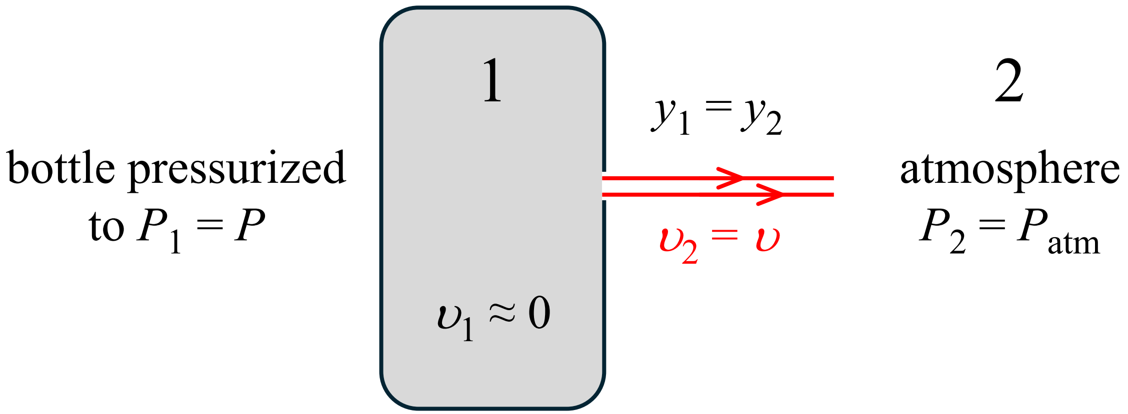
integrate assuming isothermal $\Rightarrow \frac{RT}{M} \ln P + \frac{1}{2}\upsilon^2 + gy = \text{const} \quad -(4)$

sub (3) $\Rightarrow \frac{P}{\rho} \underbrace{\ln P} + \frac{1}{2}\upsilon^2 + gy = \text{const}$

extra factor due to compressibility

Bernoulli equation for isothermal gas

$$\frac{RT}{M}\ln P+\frac{1}{2}\upsilon^2+gy=\text{const} \quad -(4)$$



$$(4) \quad \Rightarrow \quad \frac{RT}{M}\ln P=\frac{RT}{M}\ln P_{\text{atm}}+\frac{1}{2}\upsilon^2$$

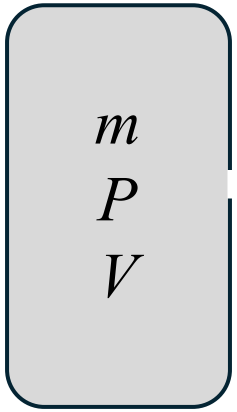
$$\therefore \ \upsilon=\sqrt{\frac{2RT}{M}\ln\frac{P}{P_{\text{atm}}}} \quad -(5)$$

Correctly implies $\upsilon \to 0$ as $P \to P_{\text{atm}}$.

$$PV = nRT = \frac{m}{M}RT \quad -(2)$$

$$\rho = \frac{M}{RT}P \quad -(3)$$

initially

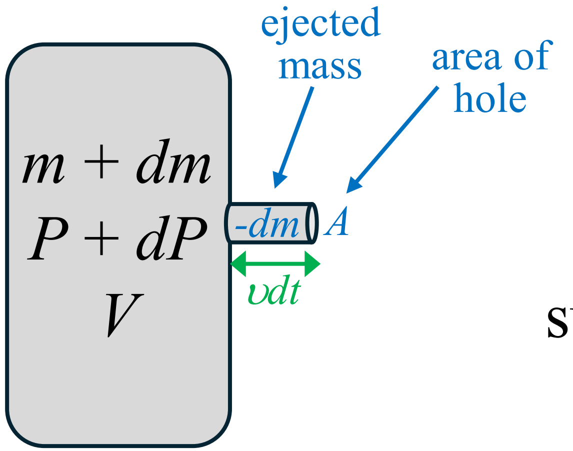


$$(2) \Rightarrow m = \frac{PVM}{RT} \Rightarrow dm = \frac{VM}{RT}dP$$

finally

$dm < 0$

$dP < 0$



$$-dm = \rho A \upsilon dt$$

$$\text{sub (3)} \Rightarrow dm = -\frac{M}{RT}PA\upsilon dt$$

$$\text{equate red expressions} \Rightarrow VdP = -PA\upsilon dt$$

$$\div PV \Rightarrow \frac{dP}{P} = -\frac{A\upsilon}{V}dt$$

$$\upsilon = \sqrt{\frac{2RT}{M}\ln\frac{P}{P_{atm}}} \quad -(5)$$

again $\frac{dP}{P} = -\frac{A\upsilon}{V}dt$

sub (5) $\Rightarrow \frac{dP}{P\sqrt{\ln(P/P_{atm})}} = -\frac{A}{V}\sqrt{\frac{2RT}{M}}dt$

define time constant $\tau \equiv \frac{V}{A}\sqrt{\frac{M}{2RT}}$

$\therefore \int_{P_0}^{P} \frac{dP}{P\sqrt{\ln(P/P_{atm})}} = -\frac{1}{\tau}\int_0^t dt$

but $\frac{d}{dP}2\sqrt{\ln(P/P_{atm})} = \frac{1/P}{\sqrt{\ln(P/P_{atm})}}$

so that $\sqrt{\ln\frac{P}{P_{atm}}} - \sqrt{\ln\frac{P_0}{P_{atm}}} = -\frac{t}{2\tau}$

$$\therefore P = P_{atm}\exp\left[\left(\sqrt{\ln\frac{P_0}{P_{atm}}} - \frac{t}{2\tau}\right)^2\right]$$

again $P = P_{atm} \exp\left[\left(\sqrt{\ln\frac{P_0}{P_{atm}}} - \frac{t}{2\tau}\right)^2\right]$ –(6) Correctly simplifies to $P = P_0$ at $t = 0$.

Also $P = P_{atm}$ when $\sqrt{\ln\frac{P_0}{P_{atm}}} = \frac{t_f}{2\tau}$ $\Rightarrow$ $t_f = 2\tau\sqrt{\ln\frac{P_0}{P_{atm}}}$.

<u>Experiment</u>: Pump up a 2 L soda bottle to 3 atm in a room at 20°C and let the air flow out of a 0.5 mm diameter hole.

$$R = 8.314 \text{ J K}^{-1}\text{ mol}^{-1}$$

$$T = 293 \text{ K}$$

$$M = 0.78\overbrace{(28 \text{ g/mol})}^{N_2} + 0.21\overbrace{(32 \text{ g/mol})}^{O_2} + 0.01\overbrace{(40 \text{ g/mol})}^{Ar}$$

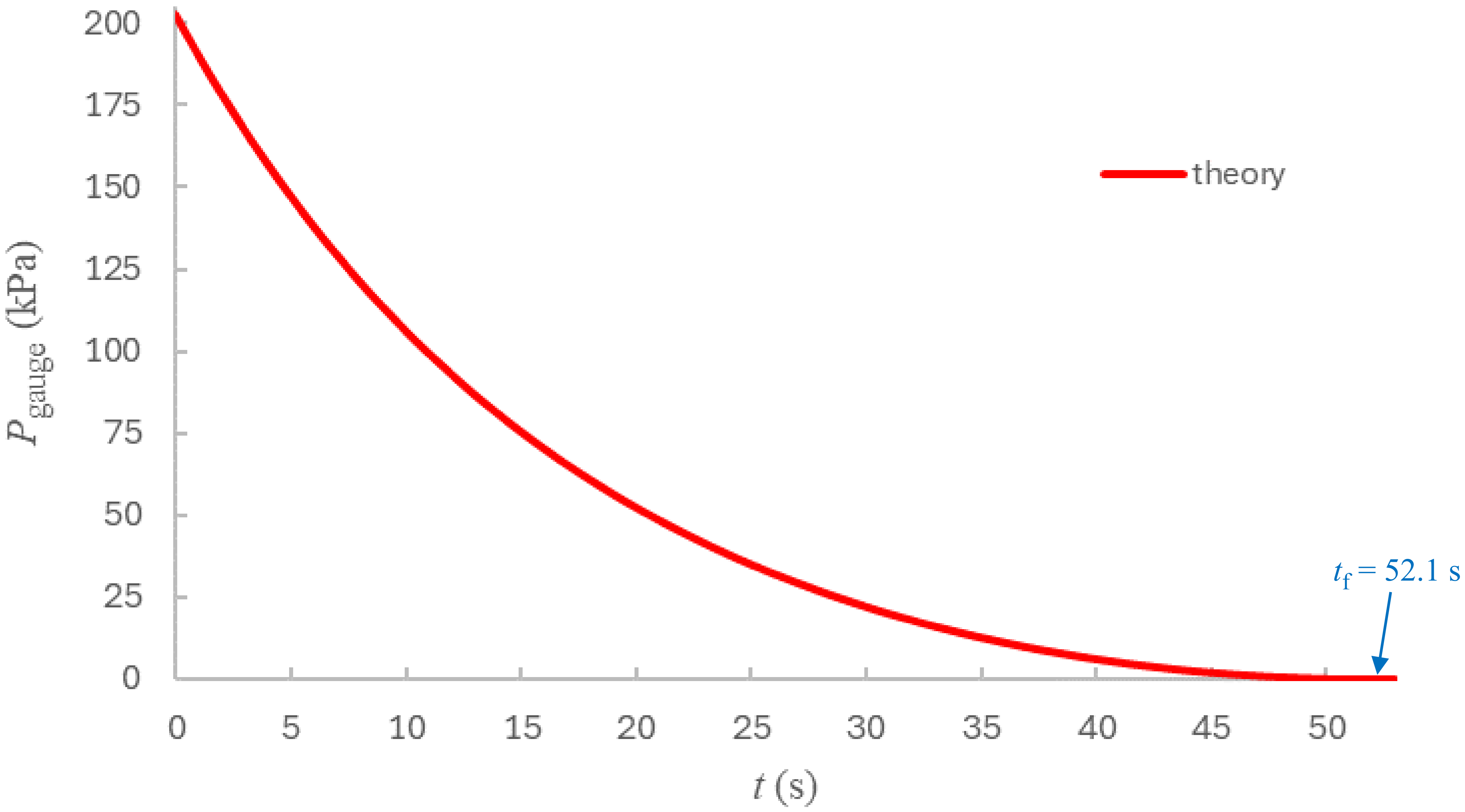
$$V = 2\times10^{-3}\text{ m}^3$$

$$D = 0.5 \text{ mm}$$

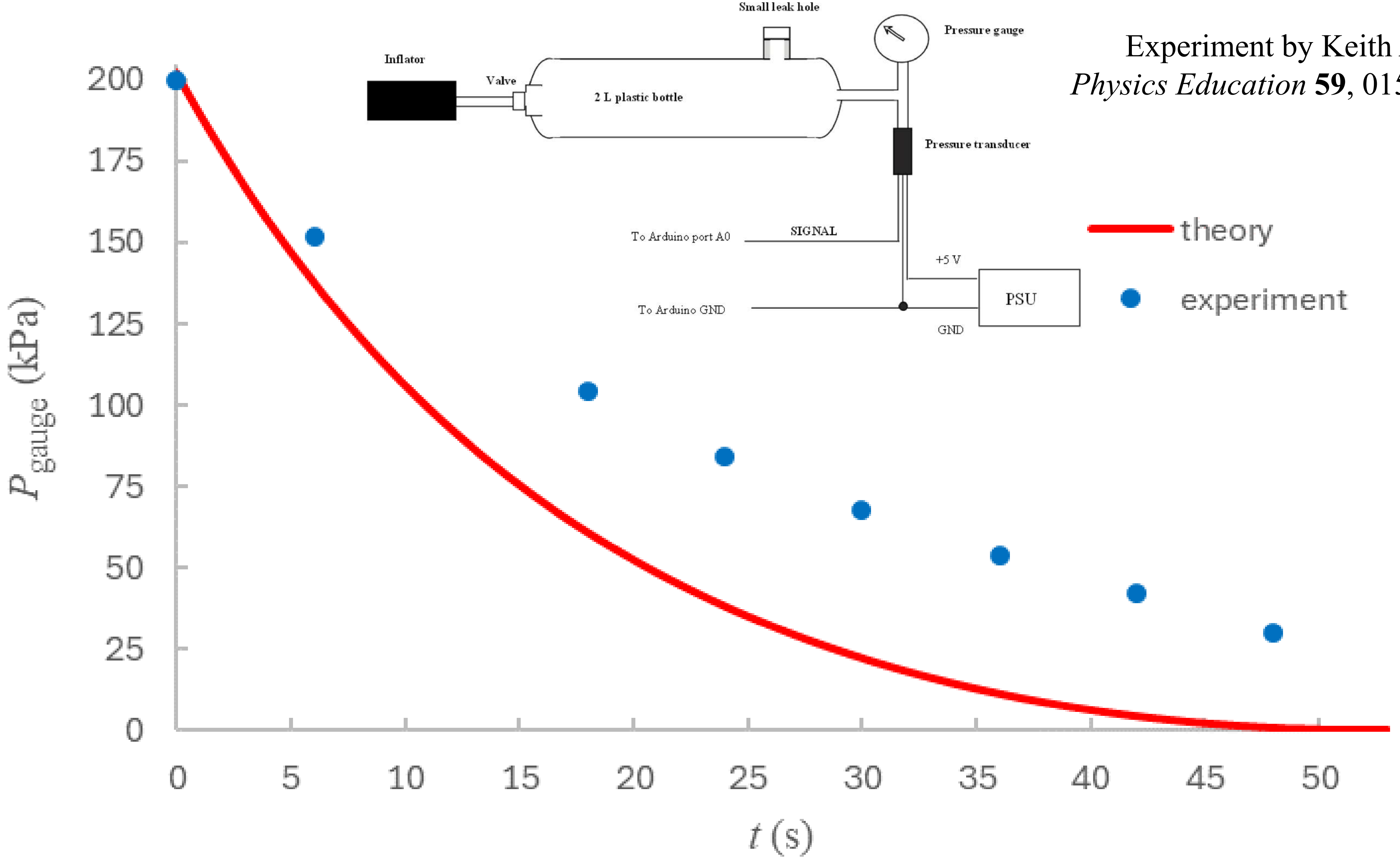
$$A = 0.25\pi D^2$$

$$P_0 = 3 \text{ atm}$$

$$P_{atm} = 101.3 \text{ kPa}$$


200
175
150
125
100
75
50
25
0
P_{gauge} (kPa)
theory
t_f = 52.1 s
0
5
10
15
20
25
30
35
40
45
50
t (s)

Experiment by Keith Atkin in *Physics Education* **59**, 015035 (2024).



To improve the fit, adjust two parameters to match those used by Atkin:

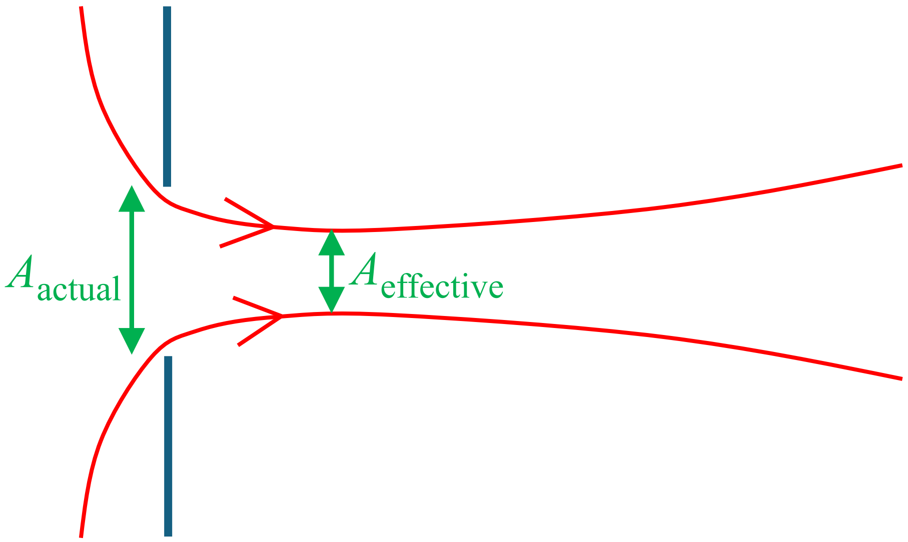
M = 30 g/mol (instead of 29.0 g/mol)

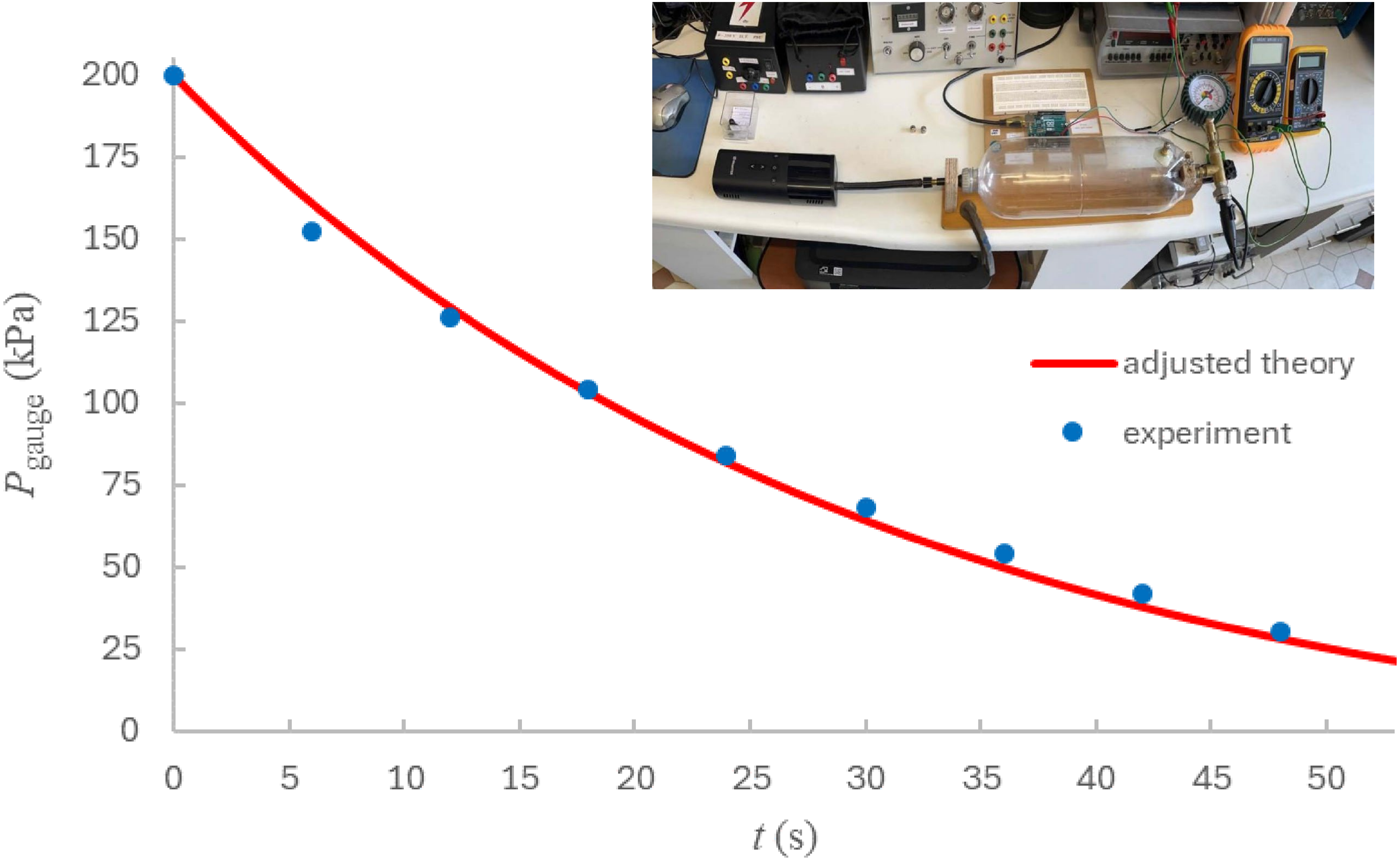
P_{atm} = 100 kPa (instead of 101.3 kPa)

and reduce the *effective* hole diameter to:

D = 0.38 mm (instead of 0.50 mm).

This corresponds to reducing the hole area to 58% of its actual area, which is the *discharge coefficient* resulting from the *vena contracta* and flow losses at the hole.




200
175
150
125
100
75
50
25
0
P_{gauge} (kPa)
0
5
10
15
20
25
30
35
40
45
50
t (s)
adjusted theory
experiment

Application: Consider the depressurization of a spacecraft struck by a micrometeroid. Does the crew only have seconds to survive? Unfortunately, if we set the initial pressure to $P_0 = 1$ atm and the ambient surroundings to $P_{atm} = 0$, our key equation fails:

$$P = P_{atm} \exp\left[\left(\sqrt{\ln \frac{P_0}{P_{atm}}} - \frac{t}{2\tau}\right)^2\right]$$

There are two issues. First, heat cannot be transferred from vacuum and so the gas expanding into the ambient surroundings cannot remain isothermal. Instead it must cool down and we instead need the Bernoulli equation for an *adiabatic* gas. Secondly, once the pressure inside the vessel gets sufficiently low (when the mean free path becomes larger than the hole diameter) we need to model the flow as an *effusion process* (as described in standard thermodynamics textbooks). Such a lab experiment has been performed and analyzed in AJP **37**, 39 (1969). The result is it takes a long 18 minutes for the pressure to fall to 0.01 atm for a 3.5 L vessel with a 0.37 mm actual diameter hole.

Thoughts or questions?

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where you can find the paper:

Comment on “The spacecraft decompression problem,”
Physics Education 59, 038003 (2024)