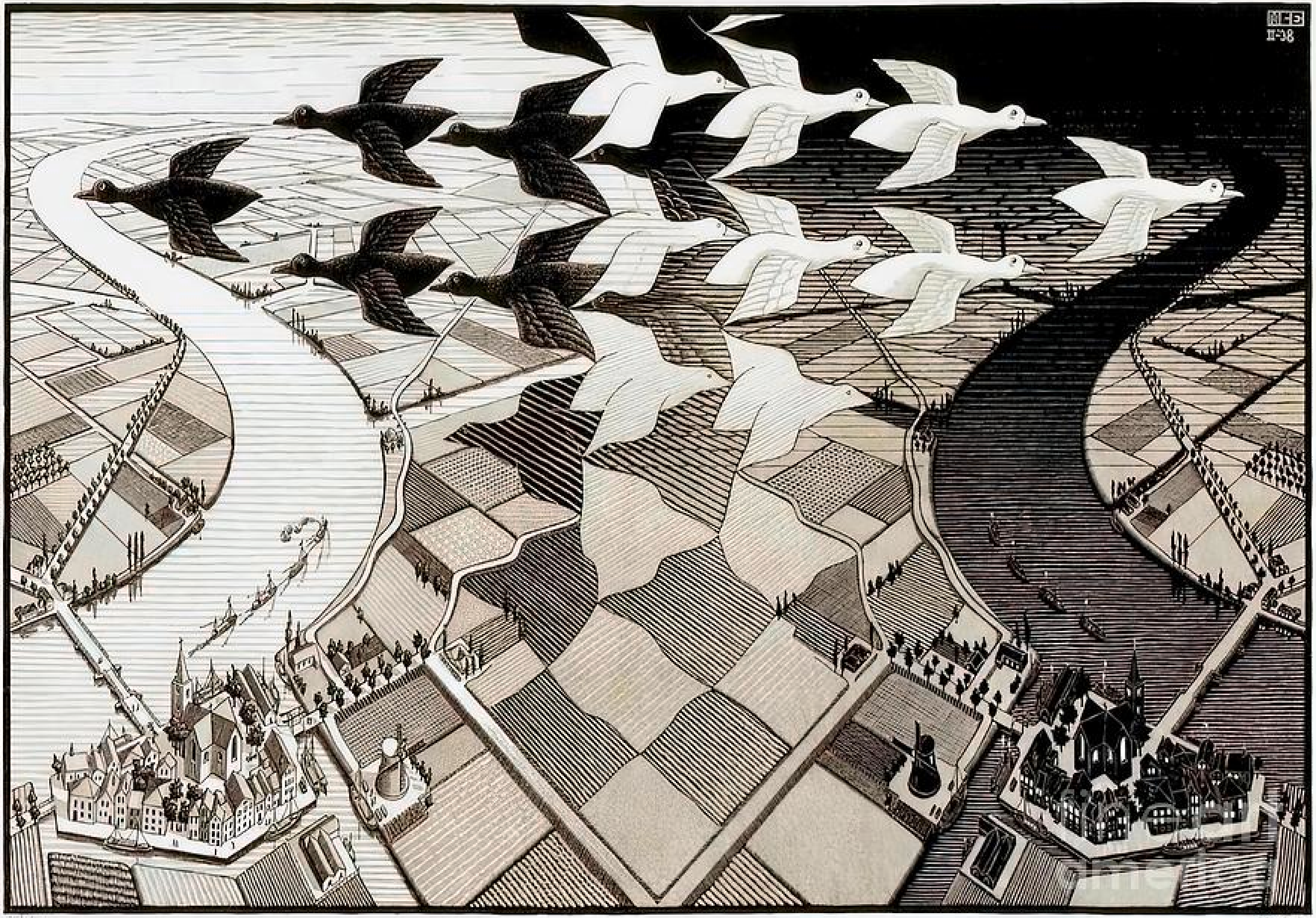
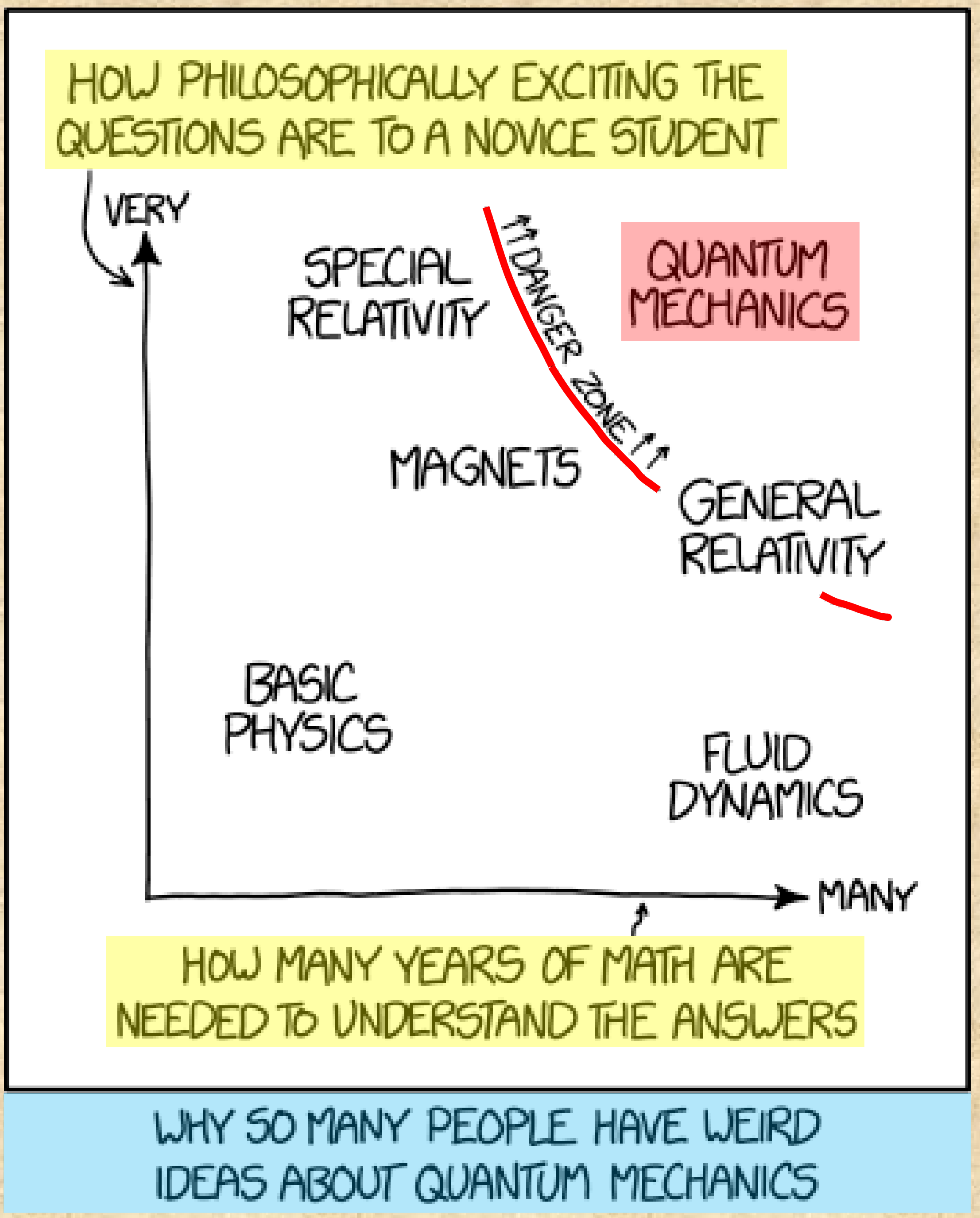
Superposition of Quantum Mechanics and SCALE-UP in the Classroom

Gerald Feldman[1], Rahul Simha[2], Denis F. Cioffi[1]

[1]Dept. of Physics [2]Dept. of Computer Science

THE GEORGE WASHINGTON UNIVERSITY
WASHINGTON DC




HOW PHILOSOPHICALLY EXCITING THE QUESTIONS ARE TO A NOVICE STUDENT
VERY
SPECIAL RELATIVITY
!! DANGER ZONE !!
QUANTUM MECHANICS
MAGNETS
GENERAL RELATIVITY
BASIC PHYSICS
FLUID DYNAMICS
MANY
HOW MANY YEARS OF MATH ARE NEEDED TO UNDERSTAND THE ANSWERS
WHY SO MANY PEOPLE HAVE WEIRD IDEAS ABOUT QUANTUM MECHANICS

A vexing question…

Quantum Mechanics is the upper-level undergraduate course that is often the most challenging in the core curriculum for physics majors. While active-learning pedagogical approaches such as SCALE-UP are gaining traction in introductory physics courses at institutions throughout the country, most upper-level classes are still largely taught in a conventional lecture mode.

One could ask the following question (which I have been asked in the past):

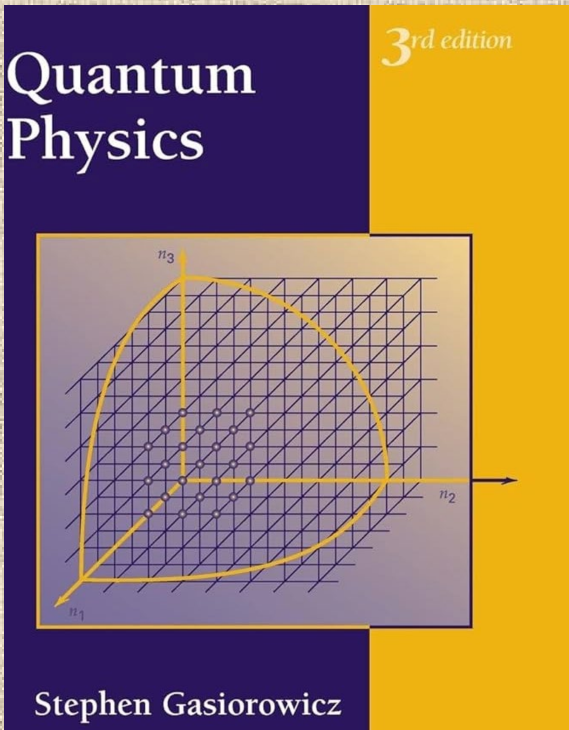
"If we expect **first-year students** to prepare in advance for class and to work together on **introductory-level problems and exercises** in a SCALE-UP collaborative group-learning classroom, then why don't we correspondingly expect **third-year students** to do the same in their **upper-level classes**?"

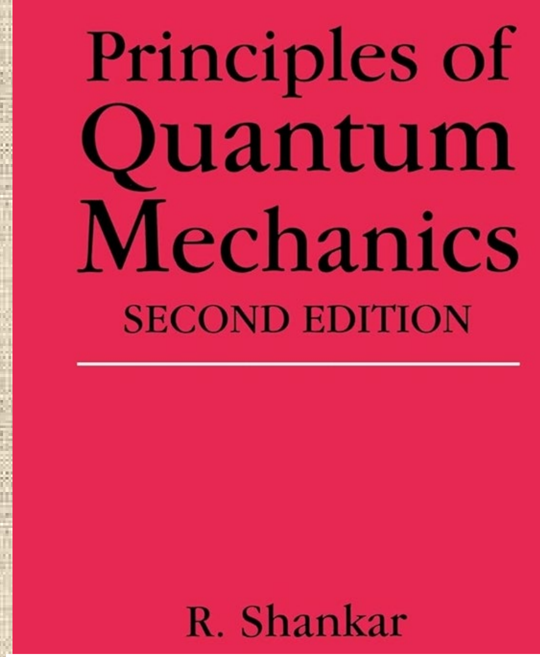
Basics of the Course

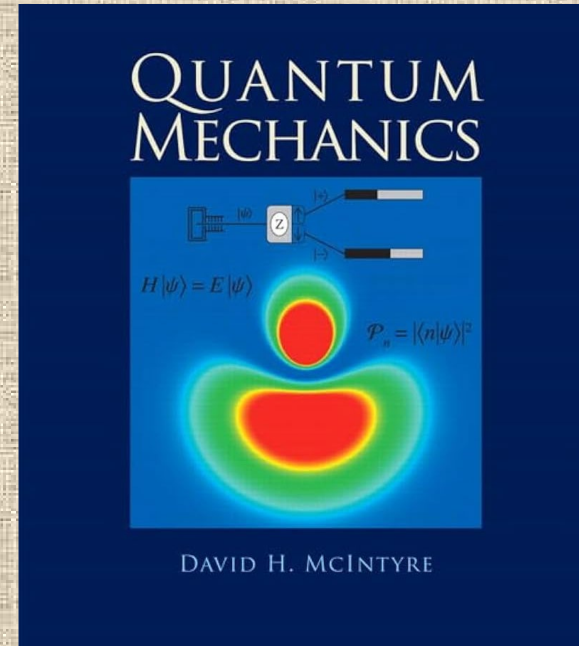
Quantum Mechanics

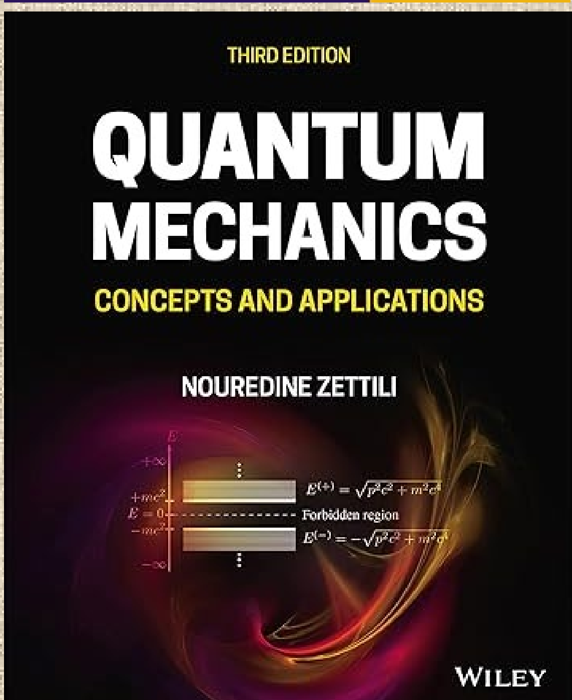
- **Modern Physics**: usually in 2nd undergraduate year
  - Special Relativity
  - Quantum Physics
  - Atomic Physics
  - Selected topics (nuclear, solid state, molecular, astro, etc.)

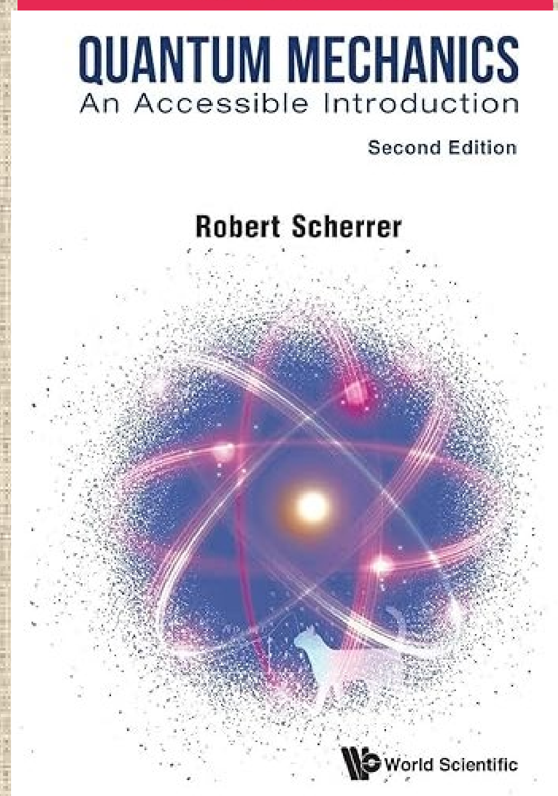
- **Quantum Mechanics**: usually in 3rd or 4th year
  - can be a one-semester or two-semester course
  - QM course at George Washington University
    - one semester – class scheduled twice a week (75 minutes)
    - primary instructor (GF), Friday tutor (RS), consultant (DFC)
    - students: six female physics majors
    - three exams (70%), five quizzes (20%), weekly homework (10%)
    - supplementary session on Friday (60-90 minutes)

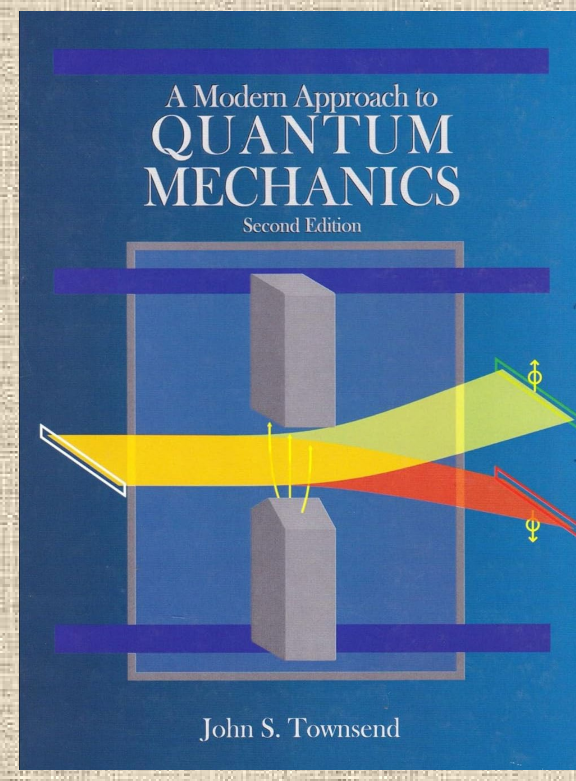

3rd edition
Quantum
Physics
Stephen Gasiorowicz

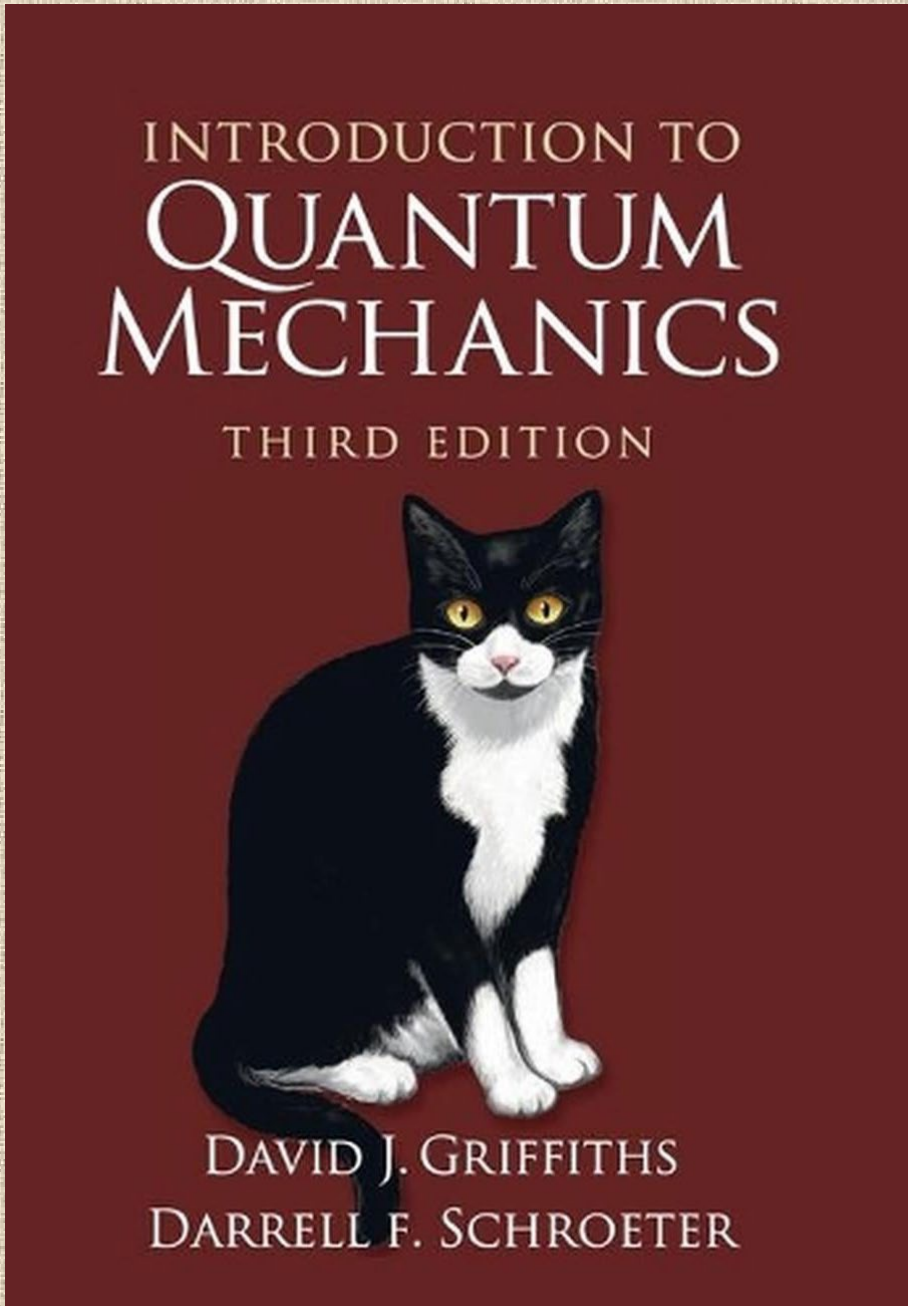

Principles of
Quantum
Mechanics
SECOND EDITION
R. Shankar

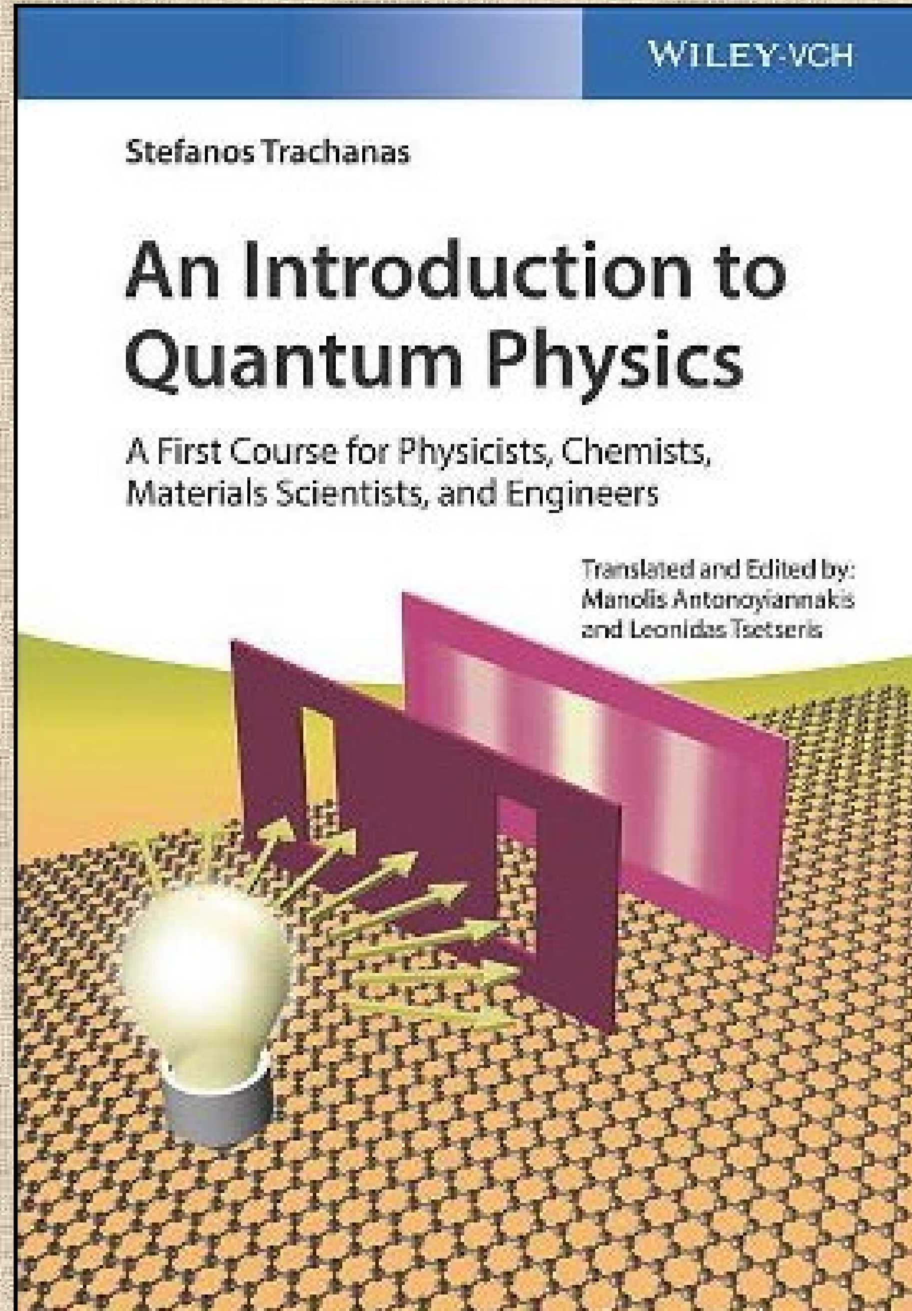

QUANTUM
MECHANICS
DAVID H. MCINTYRE


THIRD EDITION
QUANTUM
MECHANICS
CONCEPTS AND APPLICATIONS
NOUREDINE ZETTILI
Forbidden region
WILEY


QUANTUM MECHANICS
An Accessible Introduction
Second Edition
Robert Scherrer
World Scientific


A Modern Approach to
QUANTUM
MECHANICS
Second Edition
John S. Townsend


INTRODUCTION TO
QUANTUM
MECHANICS
THIRD EDITION
DAVID J. GRIFFITHS
DARRELL F. SCHROETER


WILEY-VCH
Stefanos Trachanas
An Introduction to
Quantum Physics
A First Course for Physicists, Chemists,
Materials Scientists, and Engineers
Translated and Edited by:
Manolis Antonoyiannakis
and Leonidas Tsetseris

A. Buzzell, T. Atherton, R. Barthelemy
(Phys. Rev. PER 2025)
Quantum Mechanics

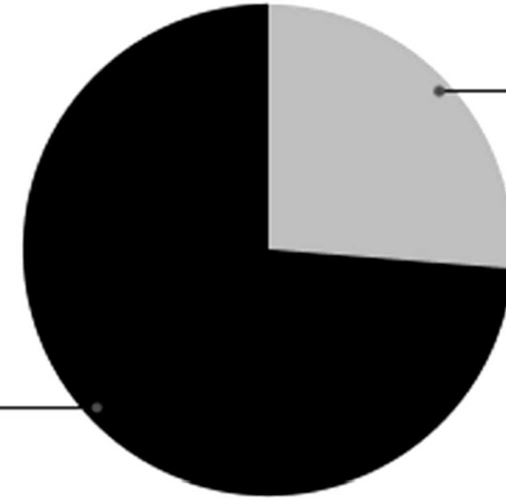


FIG. 6. Spin-first versus position-first paradigm used in the quantum mechanics courses of 99 institutions.

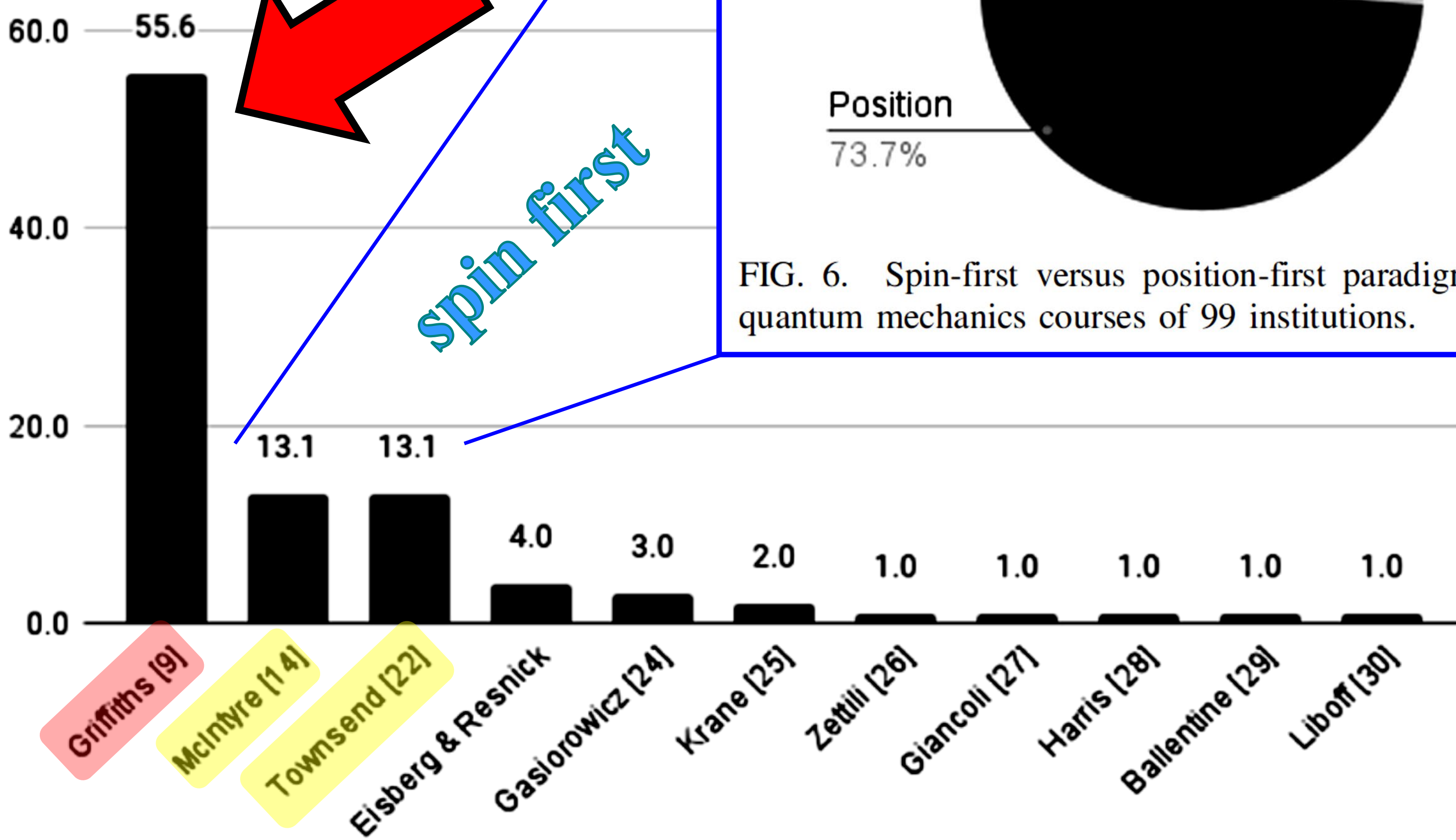
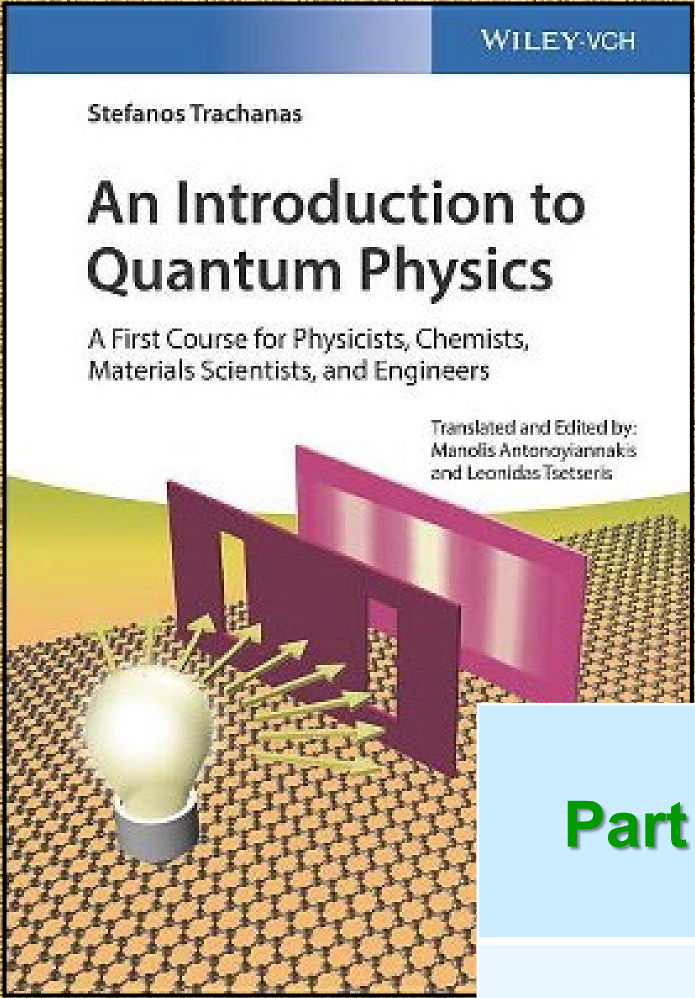


FIG. 7. Textbooks used by quantum mechanics courses at 99 institutions.

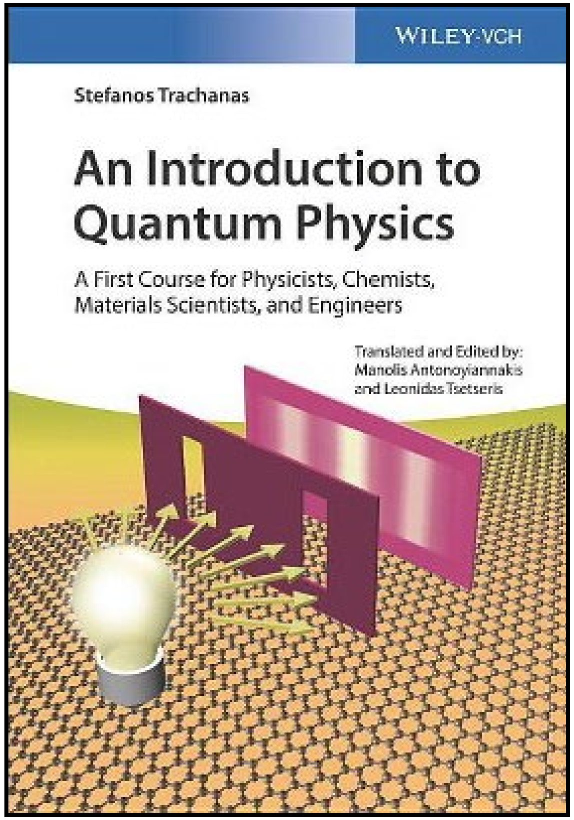


Part I	Fundamental Principles
Part II	Simple Quantum Systems
Part III	Quantum Mechanics in Action: The Structure of Matter

Chap. 1	Principle of Wave-Particle Duality
2	Schrödinger Equation and its Statistical Interpretation
3	Uncertainty Principle
Chap. 4	Square Potentials I: Discrete Spectra – Bound States
5	Square Potentials II: Continuous Spectra – Scattering States
6	The Harmonic Oscillator
7	Polynomial Method: Systematic Theory and Applications
8	Hydrogen Atom I: Spherically Symmetric Solutions
9	Hydrogen Atom II: Solutions with Angular Dependence
10	Atoms in a Magnetic Field and the Emergence of Spin
11	Identical Particles and the Pauli Principle
Chap. 12	Atoms and the Periodic Table
13	Molecules I: Elementary Theory of Chemical Bonds
14	Molecules II: The Chemistry of Carbon
15	Solids: Conductors, Semiconductors, Insulators
16	Matter and Light: Interactions of Atoms with EM Radiation

It is an unusual book. All the formulas and numbers and tables that you find in any other textbook on the subject are there. This level of systematic detail is important; one does expect a textbook to contain a complete treatment of the subject and to serve as a reference for key results and expressions. But there are also many wonderful insights that I have not found elsewhere, and numerous elaborate discussions and explanations of the *meaning* of the formulas, a crucial ingredient for developing an *understanding* of quantum physics.

Efthimios Kaxiras



Activities in the Course


WELL, WHEN WE OBSERVE THEM, THEY BECOME AMBER *PARTICLES* OF GRAIN.

ConcepTest 2 Energy Probability

For the following arbitrary wave function, what is the probability of measuring E_1 or E_2?

$$\psi = \frac{1}{4}\psi_1 + \frac{3}{4}\psi_2$$

1. 1/4 or 3/4
2. 1/16 or 9/16
3. 1 or 3
4. 1/10 or 9/10
5. 1/2 or √3 / 2

ConcepTest 3 Energy Probability

For the following arbitrary wave function, what is the probability of measuring E_3?

$$\psi = 2\psi_1 + 3\psi_2 + 6\psi_3$$

1. 0.167
2. 0.735
3. 0.028
4. 0.545
5. 0.438

Consider a particle whose wave function is given by:

$$\psi = N(3\psi_1 + 4\psi_2)$$

where the energy eigenvalues are $E_1 = 5$ and $E_2 = 10$.

(a) Find the normalization constant. $N = 1/5$

(b) Calculate the mean energy $\langle E \rangle$.

$$\langle E \rangle = P_1 E_1 + P_2 E_2 = \frac{9}{25} \cdot 5 + \frac{16}{25} \cdot 10 = \frac{41}{5} = 8.2$$

(c) Calculate the energy uncertainty ΔE.

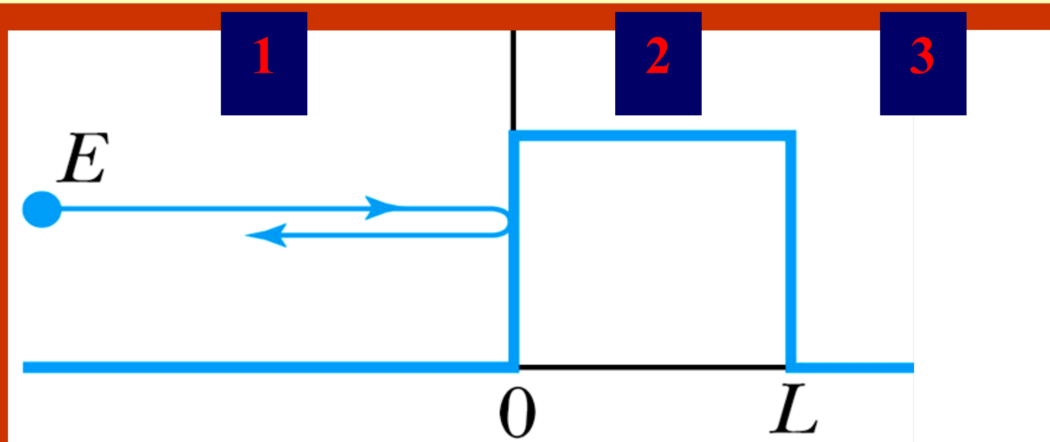
$$\langle E^2 \rangle = P_1 E_1^2 + P_2 E_2^2 = \frac{9}{25} \cdot 25 + \frac{16}{25} \cdot 100 = 73$$

$$(\Delta E)^2 = \langle E^2 \rangle - \langle E \rangle^2 = 73 - 67.24 = 5.76 \quad \Rightarrow \quad \Delta E = 2.4$$

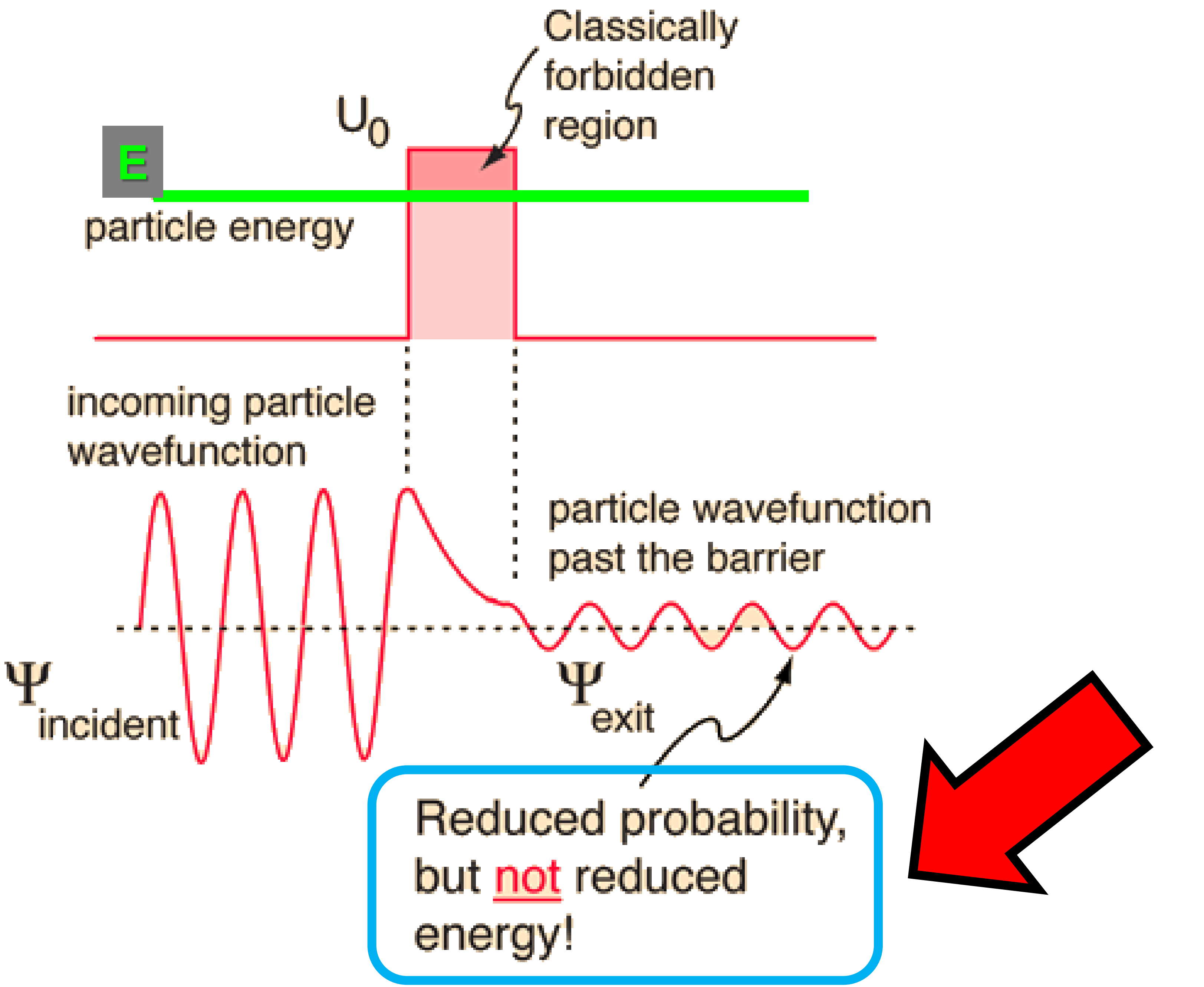
ConcepTest 4 Tunneling

How does the energy of the particles that have tunneled through the barrier compare to their initial energy?

1. it is lower than the initial
2. it is higher than the initial
3. it is the same as the initial
4. it is zero in all cases



The wave number *k* is the <u>same</u> in regions 1 and 3, so the energy of the particle is the same when the particle emerges from the barrier. **But what *does* change for the particle?**


Classically
forbidden
region
U_0
E
particle energy
incoming particle
wavefunction
particle wavefunction
past the barrier
$\Psi_{incident}$
Ψ_{exit}
Reduced probability,
but not reduced
energy!

Antje Kohnle
School of Physics/Astronomy
University of St Andrews

The Quantum Mechanics Visualisation Project

SORT

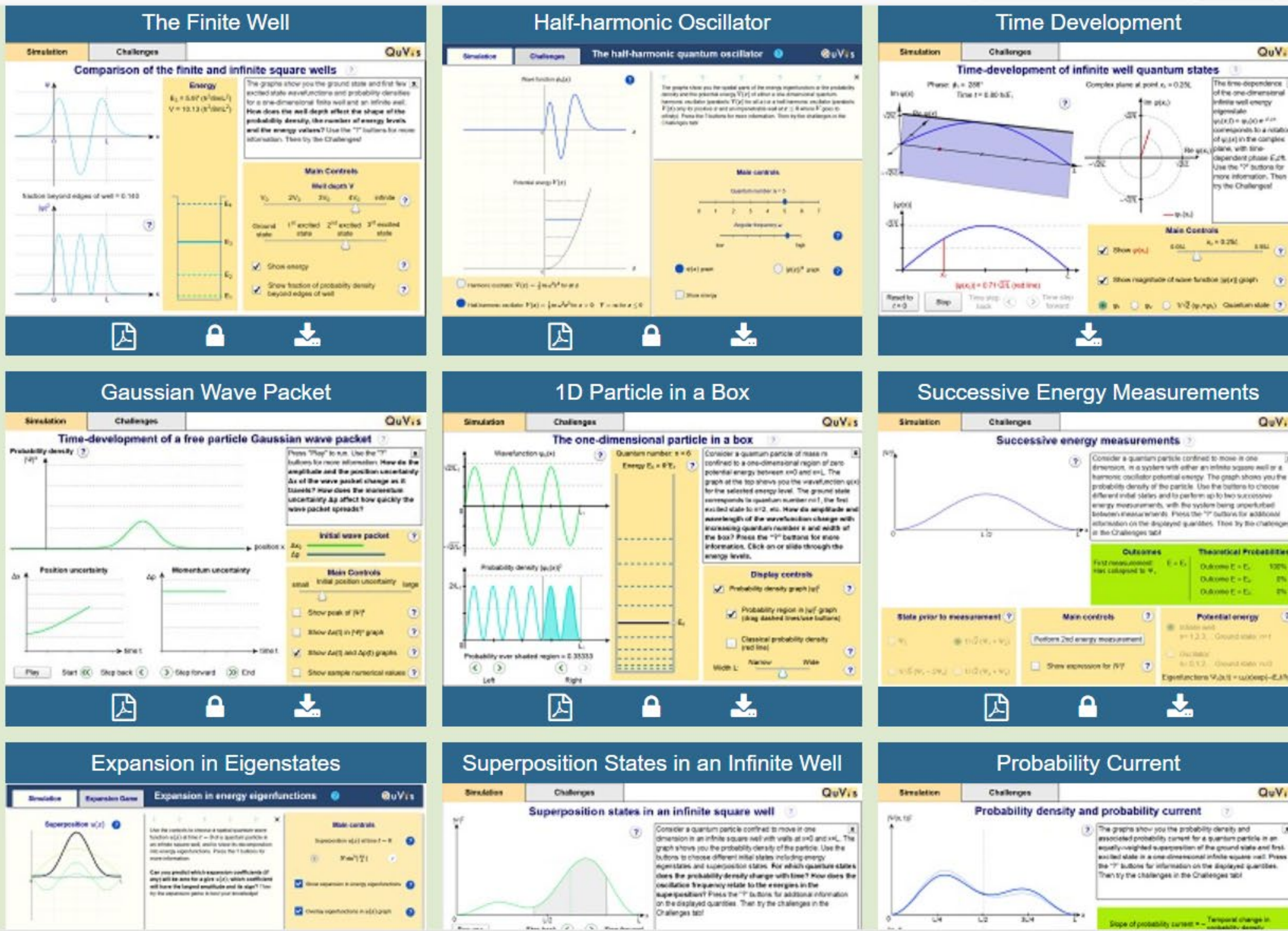
Newest | Alphabetical

TOPIC

- All Topics
- Classical systems
- Single photon experiments
- Linear algebra
- ✓ One dimensional potentials
- Two dimensional potentials
- Multiple particles
- Spin angular momentum
- Entanglement
- Fundamental concepts
- Quantum information

LEVEL

- All Levels
- ✓ Introductory
- Advanced



$$\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) \qquad c_n = \int_{-\infty}^{+\infty} \psi_n^*(x)\,\psi(x)\,dx$$

Simulation | Expansion Game

Expansion in energy eigenfunctions

QuVis

$u(x)$ and $\sum_n c_n u_n(x)$

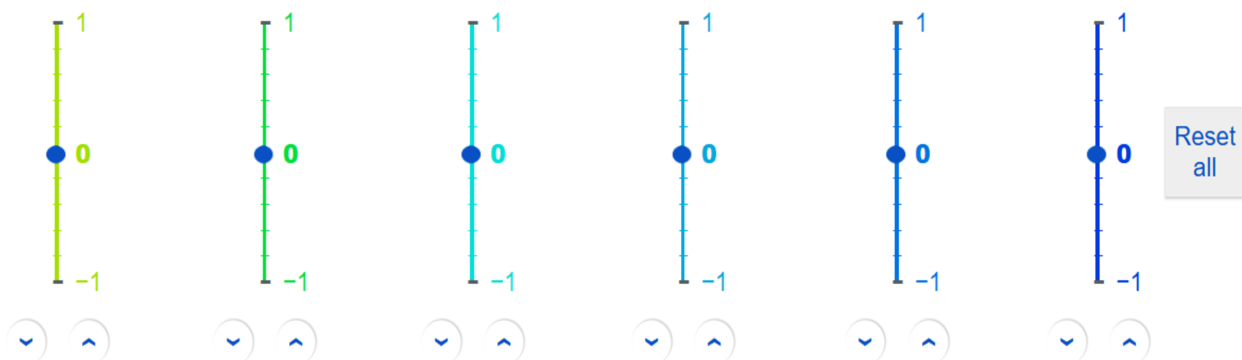
individual $c_n u_n(x)$

Your score: 0/100

Match the coefficients for the first six terms of the expansion for $u(x)$ as closely as possible using the sliders below, or the arrows for more fine control. **Coefficients that should be zero must be set to zero exactly!**

Submit | Hint

$c_1 = 0.00$ $c_2 = 0.00$ $c_3 = 0.00$ $c_4 = 0.00$ $c_5 = 0.00$ $c_6 = 0.00$



1 2 3 4

Infinite Square Well

$$\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x) \qquad c_n = \int_{-\infty}^{+\infty} \psi_n^*(x)\, \psi(x)\, dx$$

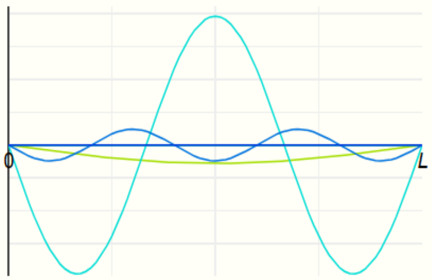
Simulation | Expansion Game

Expansion in energy eigenfunctions

QuVis

$u(x)$ and $\sum_n c_n u_n(x)$

individual $c_n u_n(x)$



1 2 3 4

Your score: 0/100

Match the coefficients for the first six terms of the expansion for $u(x)$ as closely as possible using the sliders below, or the arrows for more fine control. **Coefficients that should be zero must be set to zero exactly!**

Submit

Hint

$c_1 = -0.14$ | $c_2 = 0.00$ | $c_3 = -0.98$ | $c_4 = 0.00$ | $c_5 = -0.12$ | $c_6 = 0.00$

1 0 −1

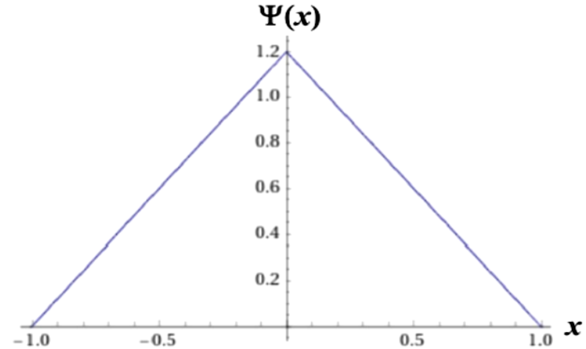
Reset all

Infinite Square Well

A particle has the following wave function restricted to the region $-1 \le x \le 1$:

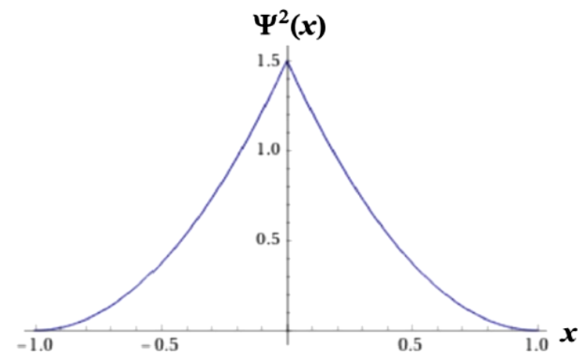
$$\begin{cases} \psi(x) = A(1-x) \longrightarrow & \text{for positive values: } 0 \le x \le 1 \\ \psi(x) = A(1+x) \longrightarrow & \text{for negative values: } -1 \le x \le 0 \end{cases}$$

a) Draw a graph of the wave function in the region $-1 \le x \le 1$.



b) Find the normalization constant A.

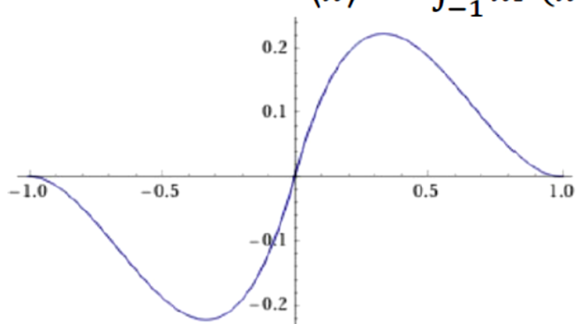
c) Draw a graph of the probability density in the region $-1 \le x \le 1$.



d) What is the probability of finding the particle in the region $-0.2 \le x \le 0.2$? Explain why the probability is relatively high for such a narrow region.

e) What is the expectation value of x?

$$\langle x \rangle = \int_{-1}^{1} xP(x)dx = \int_{-1}^{1} xA^2(1-|x|)^2 dx = 0$$



This is an <u>odd function</u> over a symmetrical interval around $x = 0$, therefore the integral will be <u>**zero**</u>.

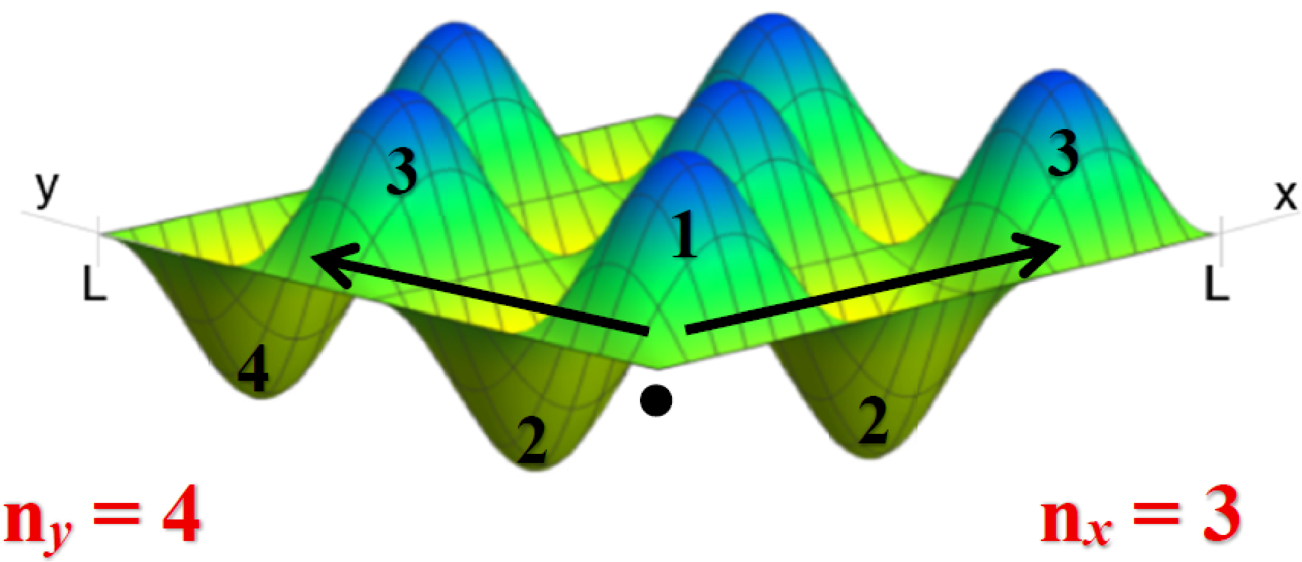
2. A particle in an infinite square well potential is in a state represented by a wave function that is a superposition of two eigenfunctions: $\psi = \sqrt{\frac{4}{5}}\psi_2 + \sqrt{\frac{1}{5}}\psi_3$.

What is the expectation value of the energy $\langle E \rangle$ for this state? Write your answer in terms of the ground-state energy E_1.

$$\langle E \rangle \;=\; P_2E_2 + P_3E_3 \;=\; \frac{4}{5}(4E_1) + \frac{1}{5}(9E_1) \;=\; \frac{16}{5}E_1 + \frac{9}{5}E_1 \;=\; 5E_1$$

---

3. Consider a 2D infinite square well, with quantum numbers n_x and n_y corresponding to the x and y directions, respectively. The width of the well is L in each direction. What are the values of n_x and n_y for this particular wave function?



3. (6 pts) For a particle trapped inside an infinite square well of fixed width, what happens to the following parameters when the quantum number n increases?

a) The energy of the particle: increases stays the same decreases

b) The amplitude of the wave function: increases stays the same decreases

c) The wavelength of the particle: increases stays the same decreases

---

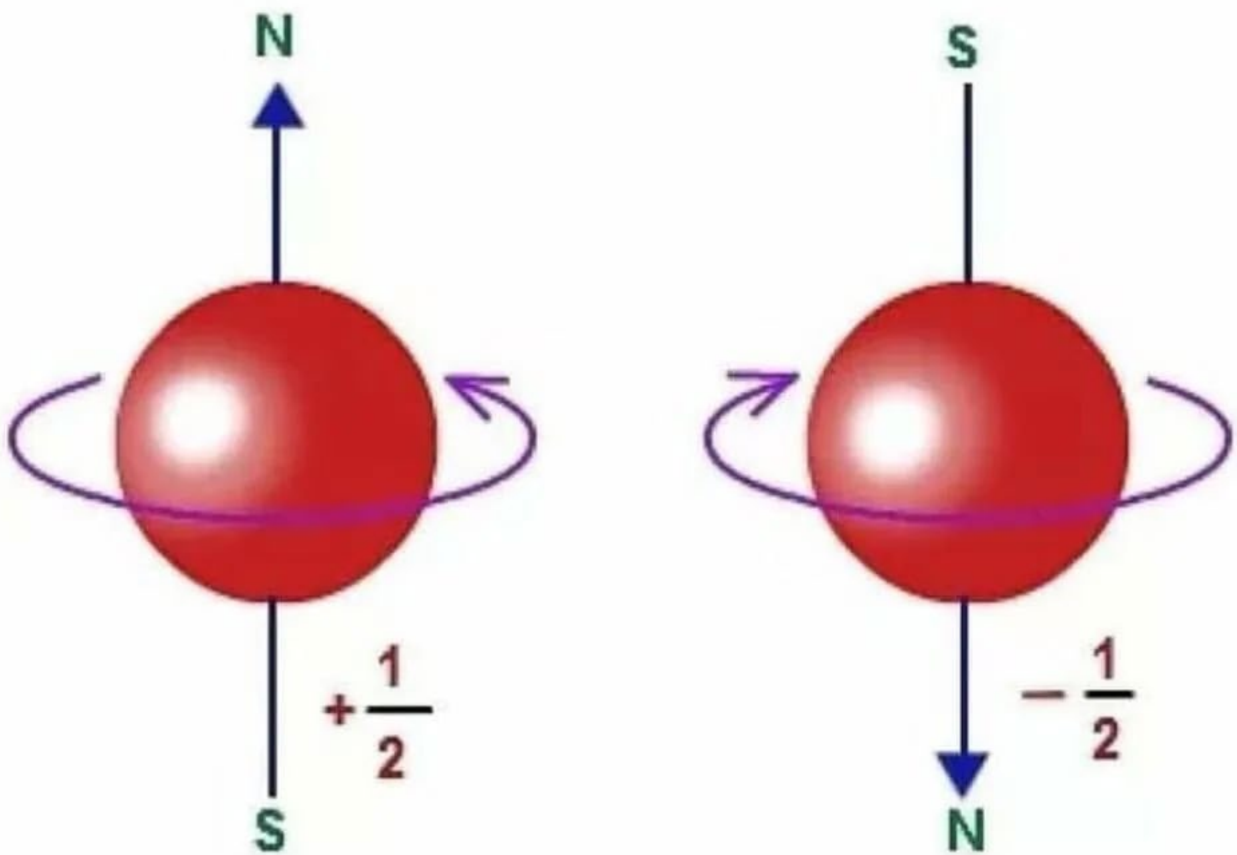
4. (6 pts) In a finite square well of fixed width and fixed depth, what happens to the wave functions or the corresponding probability densities as the various excited-state energies change?

a) As the excited-state energy E gets higher in the finite well, the fraction of the probability density that extends outside the boundary of the well:
gets greater stays the same gets lower

b) As the excited-state energy E gets higher in the finite well, the difference in energy compared to its counterpart in the infinite square well:
gets larger stays the same gets smaller

c) As the excited-state energy E gets higher in the finite well, the amplitude of the wave function:
gets larger stays the same gets smaller

Electron spin explained: imagine a ball that's rotating, except it's not a ball and it's not rotating



9. (3 pts) The state of a spin $\frac{1}{2}$ particle is described by: $\psi = N\begin{pmatrix} i \\ 2-i \end{pmatrix}$ where N is a normalization constant. What is the expectation value $\langle s_x \rangle$ of the spin along the x-axis?

a) $-\hbar$ b) $\hbar/2$ c) 0 d) $-\hbar/6$ e) $-\hbar/3$

---

10. (3 pts) For a given state vector of a spin $\frac{1}{2}$ particle, you find that the expectation value for a measurement of the mean value along the x-axis is $\langle s_x \rangle = 0$. If that is the case, then what is the probability $P_{x\uparrow}$ of measuring **spin up** along the x-axis for this state?

a) 0 b) 1/4 c) 1/2 d) 2/3 e) 1

---

11. (3 pts) For a given state vector of a spin $\frac{1}{2}$ particle, you find that the expectation value for a measurement of the mean value along the x-axis is $\langle s_x \rangle = +\frac{\hbar}{2}$. If that is the case, then what is the probability $P_{z\downarrow}$ of measuring **spin down** along the z-axis for this state?

a) 0 b) 1/4 c) 1/2 d) 2/3 e) 1

Insights from the Textbook

Wave-Particle Duality

But if the frequency is quantized in classical systems, so too will be the particle's energy, since the wave–particle duality of particles—namely, the relation $E = hf$—provides a direct link between their energy and the frequency of the corresponding wave. So if a quantum particle, say an electron, is trapped somewhere in space (e.g., in an atom or a molecule), the associated de Broglie wave will be a standing wave with quantized frequency, and therefore the energy $E = hf$ of the electron will also be quantized. As we will see shortly, energy quantization for particles that are trapped in some region of space (and thus perform confined motion) is the deepest consequence of the wave–particle duality of matter.

Probability Density

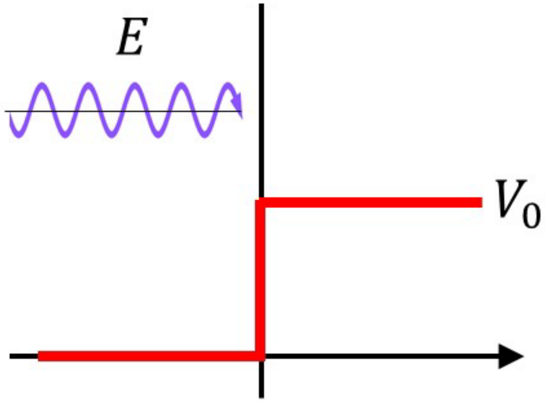
But even on purely physical grounds we can see why the proper choice for the probability density is the square of the wavefunction $|\psi(x,t)|^2$, and not any other positive quantity. Here is why. We know from classical wave theory that the energy density of a wave is given by the square of the "waving" quantity (i.e., the square of the wave amplitude). For example, in an electromagnetic wave, where the wave amplitude is given by the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, the energy density in cgs units is

$$U = \frac{1}{8\pi}(\boldsymbol{E}^2 + \boldsymbol{B}^2).$$

Integrating this expression throughout all space gives the total energy of the wave, which is a constant of motion; it is independent of time.

In matter waves, the "waving" quantity is the wavefunction $\psi(x,t)$, even if it does not describe a measurable physical disturbance in itself. Thus we can infer that, to ensure conservation of the total probability,[6] which seems to be the analog of the total energy, we should rather choose the square of the wavefunction for the probability density.

QM Reflection from a Potential



5.2.2.1 The Phenomenon of Classically Forbidden Reflection

The most notable consequence of the quantum mechanical solution we found is, surely, the possibility of reflection by the potential step even when the incident particle has the energy required to go through ($E > V_0$). This behavior is the exact opposite of the penetration into classically forbidden regions we encountered in Chapter 4. In that case, the particle entered regions where it is not allowed to be, while now it sometimes avoids entering regions where it has to enter!

the potential jump at $x = 0$ in the present case is completely analogous to the discontinuity of the refraction index n at the interface between two media, which causes an electromagnetic wave incident at the interface to be reflected.

7 It is curious that this flawed picture appears to go back all the way to the time of Schrödinger and de Broglie! (Which may partly explain its endurance, despite its falseness.) Indeed, as recounted by Felix Bloch (Section 2.2), Schrödinger himself had used this picture to obtain Bohr's quantization rules, an approach for which he was chided by Debye who characterized this way of thinking as "childish," thus prodding Schrödinger to delve deeper and come up with his eponymous wave equation. So, to a small extent, we may owe the discovery of the Schrödinger equation to this flawed picture of waves fitted along a stationary orbit.

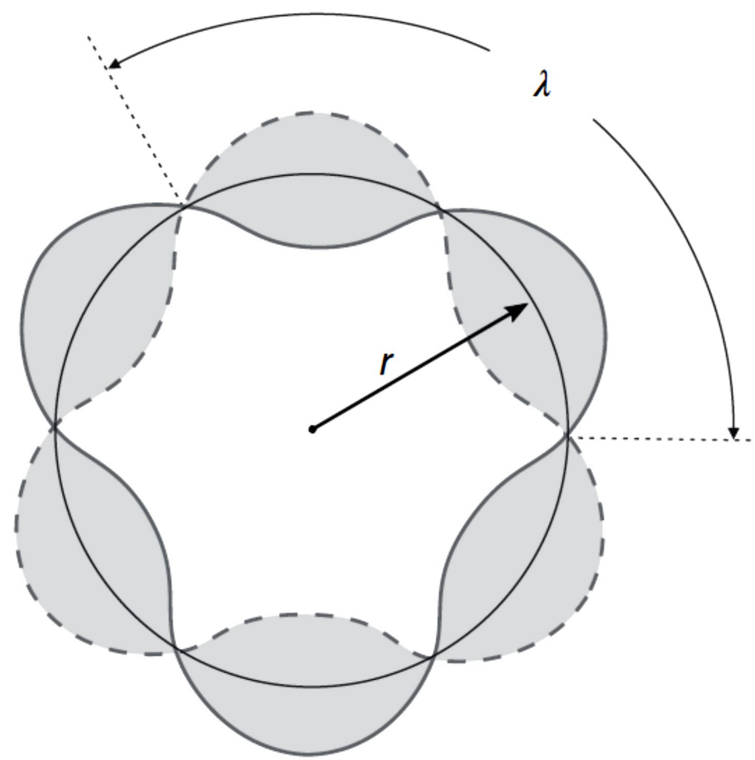


Figure 1.10 A false picture that should be discarded. Here the electron supposedly forms something like sinusoidal standing matter waves along a circle of radius *r*. But this picture is a flawed projection to three-dimensional space of the classical picture for a wave on a string. Three-dimensional waves—quantum or classical—typically fill the space and surely do not look like standing sound waves in a circular tube.

Pedagogy and Student Feedback

A. Buzzell, T. Atherton, R. Barthelemy
(Phys. Rev. PER 2025)
Modern Physics

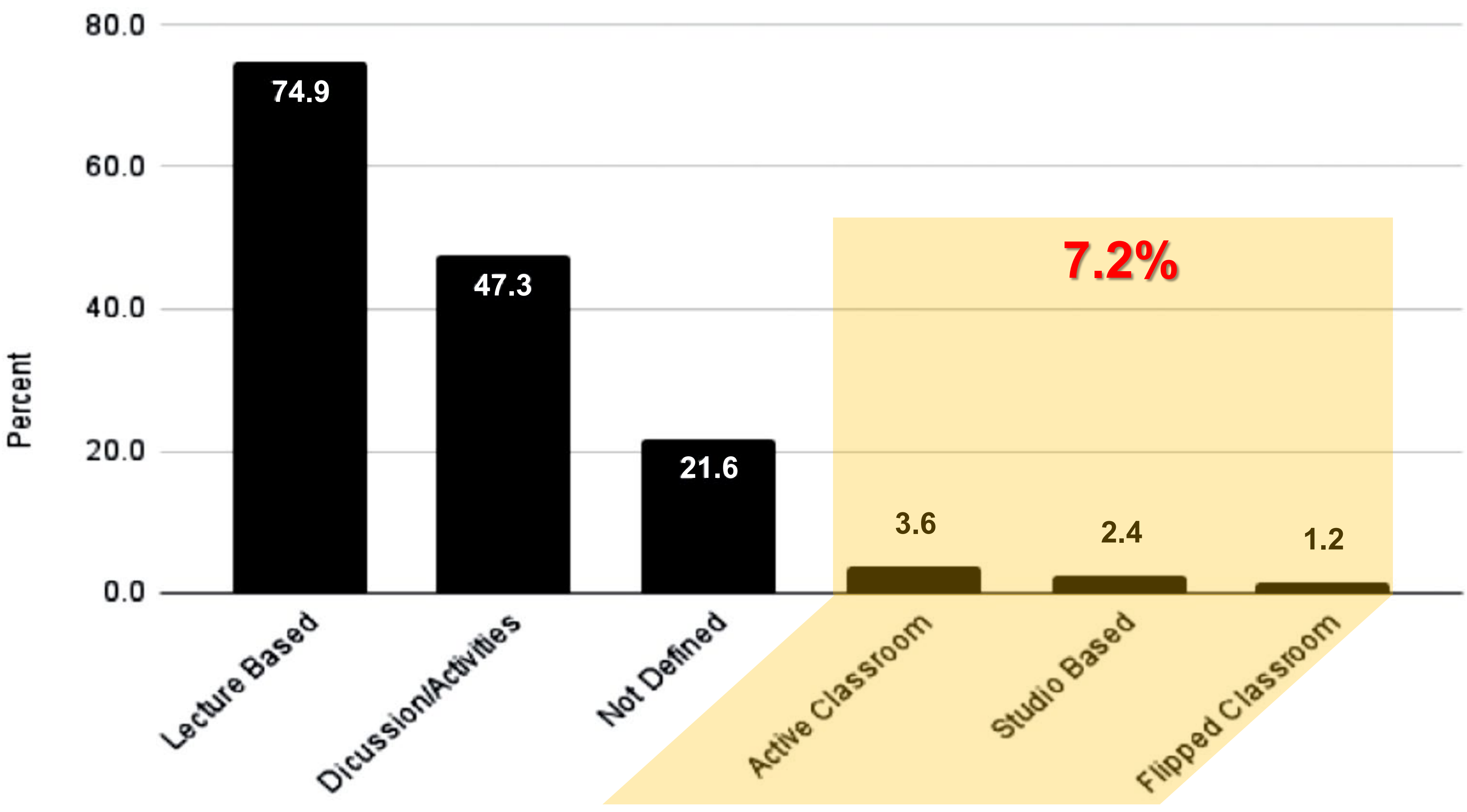
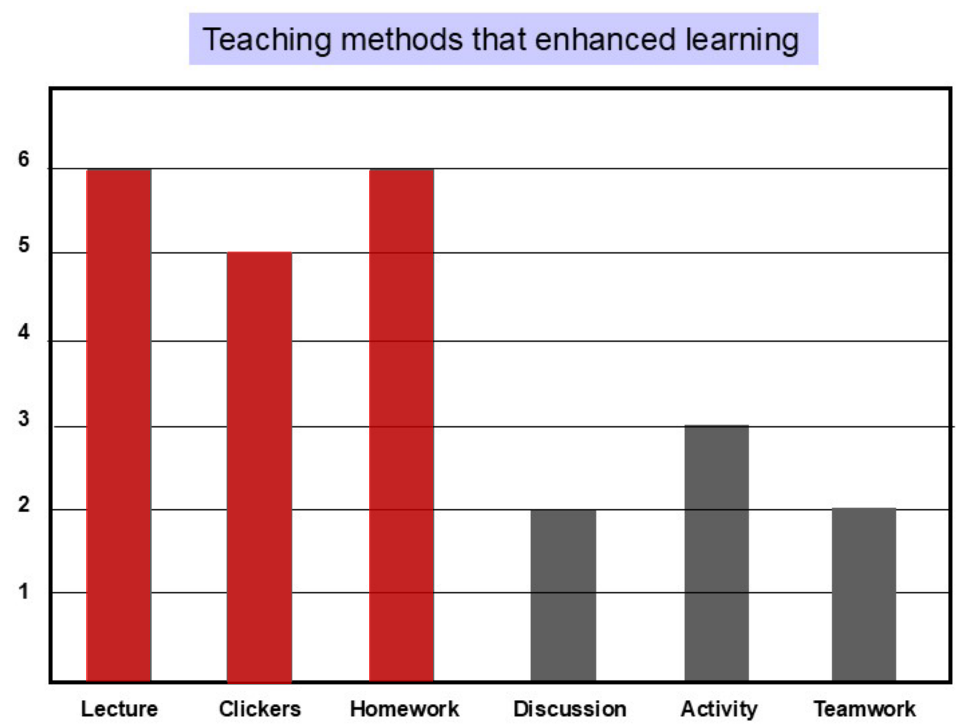
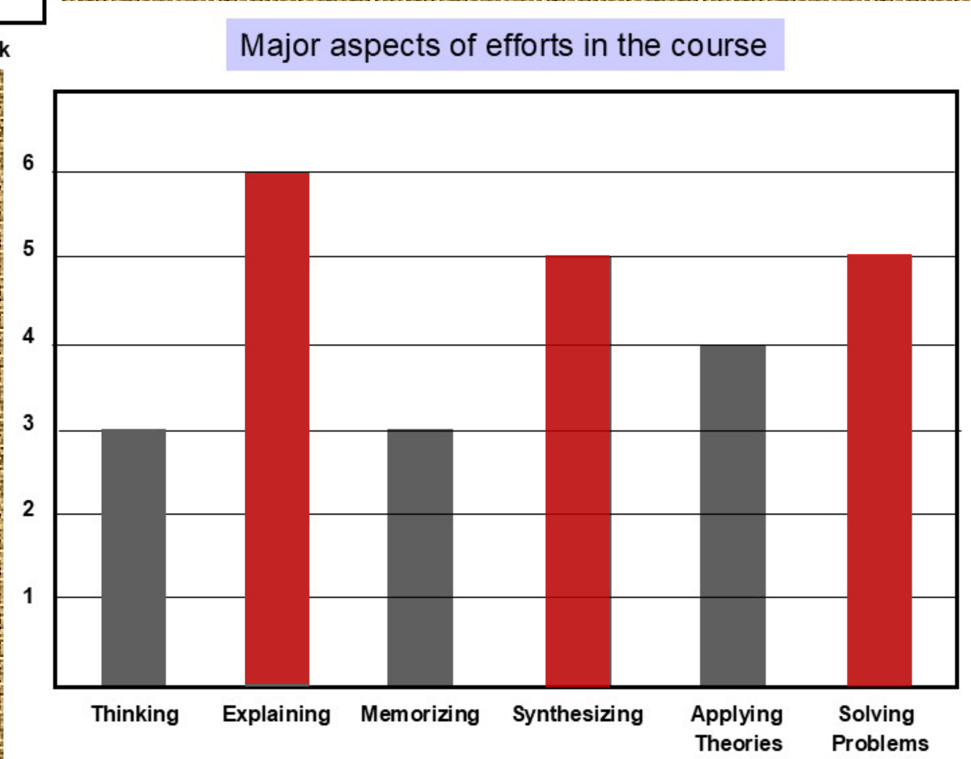


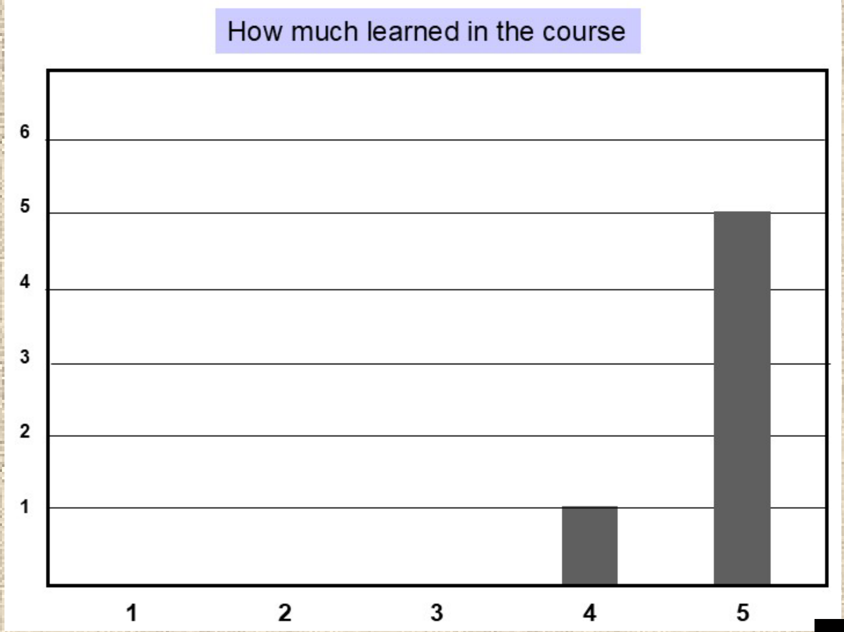
FIG. 7. Pedagogical approach utilized.

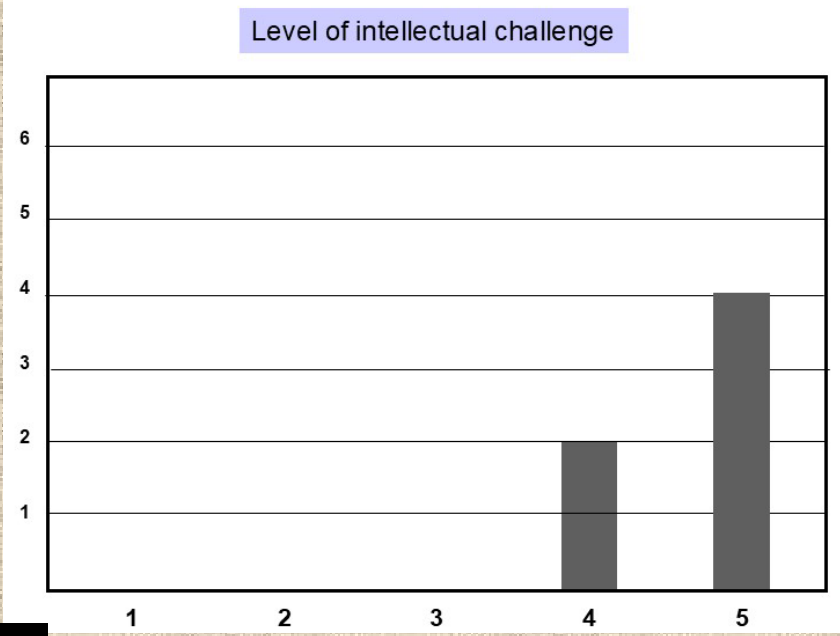
Student Feedback

N = 6

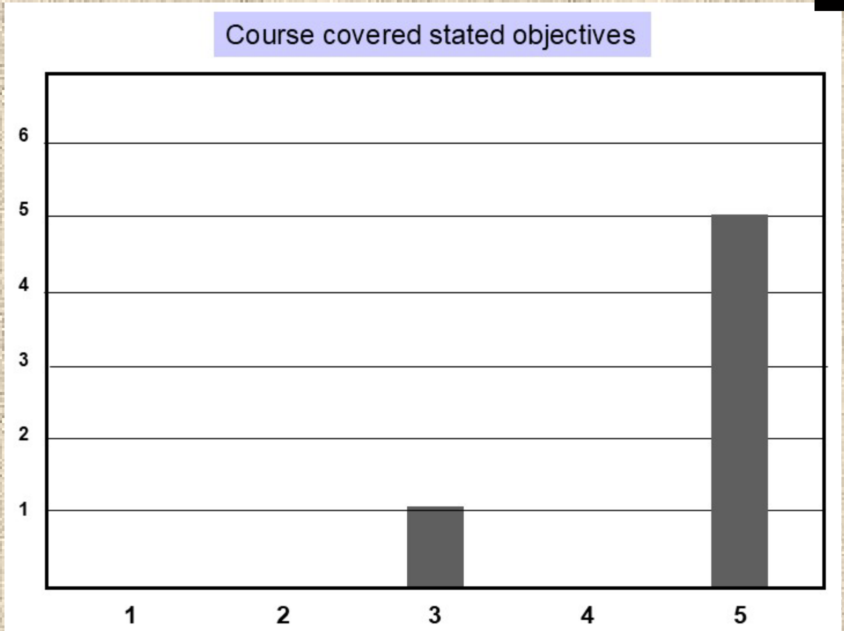


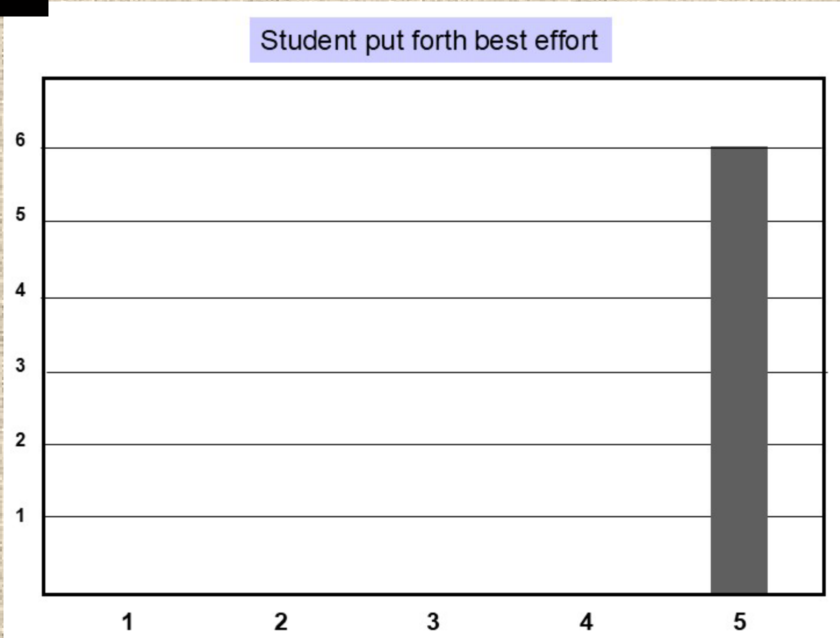



How much learned in the course
6
5
4
3
2
1
1
2
3
4
5


Level of intellectual challenge
6
5
4
3
2
1
1
2
3
4
5

N = 6


Course covered stated objectives
6
5
4
3
2
1
1
2
3
4
5


Student put forth best effort
6
5
4
3
2
1
1
2
3
4
5

Strengths of the course (from student evaluations)

- ✓ I really **appreciated how this course was taught like a modified SCALE-UP**, it was extremely useful when trying to learn a subject that feels as abstract and unintuitive as quantum mechanics.

- ✓ The **textbook was indeed interesting** and so were the quick in-class responses.

- ✓ **Really exciting and weird topics**. This was my favorite course of the whole semester. **Optional supplementary sessions were great**; I found them to be very enriching while also being low stakes, which was nice.

Pedagogical Strengths of Textbook (Google AI summary)

Intuition-First Approach: The book relies heavily on dimensional analysis and order-of-magnitude estimates to build a physical "feel" for the subject before diving into deep math.

Conceptual Focus: It treats the uncertainty principle as a central, foundational concept rather than just a mathematical consequence of commutation relations.

Broad Practical Application: Beyond theory, it includes detailed sections on quantum chemistry, chemical bonding, and the electronic structure of solids.

Expert Translation: Reviewers note the high quality of the English translation, describing the prose as "warm and inviting" and "brilliantly accessible”.

Conclusions

- Implemented SCALE-UP for upper-level QM class
  - collaborative group activities during class
  - emphasis on conceptual understanding in QM
  - development of problem-solving skills (lots of practice)
  - established a legitimate active-learning classroom!
- Basic feedback from students
  - acknowledged / appreciated SCALE-UP format of class
  - actually LIKED the textbook!
  - attended Friday supplemental sessions (voluntarily!)
- Personal conclusions (my own…)
  - this actually worked!