

# Galileo's Law of Odd Numbers

The Odd Rhythm of Free Fall & Dimensional Analysis

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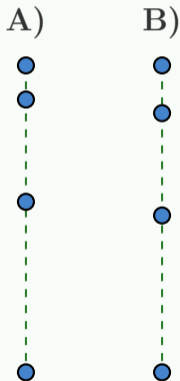
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# Part 1: The Rhythm of Free Fall

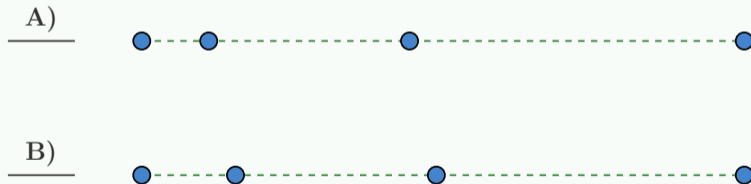
## Motion Diagrams and Historical Context

# Motion Diagrams

Introductory physics courses use motion diagrams to visualize an object's trajectory. Which diagram correctly represents the pattern of displacements for motion under constant acceleration from rest?



Uniformly accelerated motion from rest



## Observation: Diagram A

**Diagram A** is the correct representation.

If we define the distance traveled during the first time interval as our fundamental unit ( $d$ ), we can measure all subsequent displacements relative to this baseline. Evaluating the individual distances covered during consecutive, equal time intervals reveals a mathematical pattern:

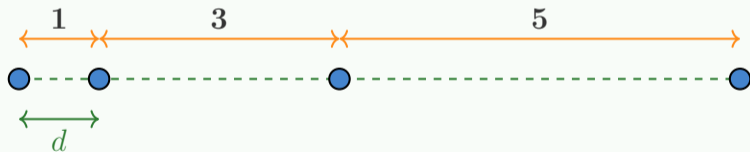
$$\Delta x_1 = 1d$$

$$\Delta x_2 = 3d$$

$$\Delta x_3 = 5d$$

$\vdots$

$$\Delta x_k = (2k - 1)d$$



# Galileo's Law of Odd Numbers

Galileo di Vincenzo Bonaiuti de' Galilei (1564–1642)

## Observation

For an object starting from rest with constant acceleration, the distance covered during equal time intervals is proportional to the odd numbers 1, 3, 5, 7... [1]



*Experimental device of Galileo. AI generated with Gemini.*

# Part 2: Dimensional Analysis

## A Unitless Perspective

# Dimensional Analysis Framework

## Variables and Dimensions

- Distance,  $d$                        $[d] = L$
- Time,  $t$                                  $[t] = T$
- Acceleration,  $a$                      $[a] = LT^{-2}$

We start by assuming a standard power-law dependence:

$$d = C \cdot a^x \cdot t^y$$

*Where  $C$  is a dimensionless constant, and the exponents  $x$  and  $y$  must be determined.*

# Deriving the Power Law

## 1. Dimensional Substitution

Substituting the dimensions yields:

$$L^1 T^0 = (LT^{-2})^x T^y$$

## 2. Exponent Resolution

Equating the exponents for each dimension:

- **Length ( $L$ ):**  $1 = x \implies \mathbf{x = 1}$
- **Time ( $T$ ):**  $0 = -2x + y$   
 $0 = -2(1) + y \implies \mathbf{y = 2}$

**Derived Relationship:**

$$d(t) = C \cdot a \cdot t^2$$

The total distance is proportional to time squared.

## Correlation with Odd Numbers

Given  $d(t) \propto t^2$ , we calculate the distance covered over successive time intervals  $\Delta t$ .

### Calculating the $j^{\text{th}}$ Interval

The distance interval  $\Delta d_j$  corresponding to the  $j^{\text{th}}$  time segment is:

$$\Delta d_j = d(j\Delta t) - d((j-1)\Delta t)$$

$$\Delta d_j \propto (j\Delta t)^2 - ((j-1)\Delta t)^2$$

$$\Delta d_j \propto \Delta t^2 [j^2 - (j^2 - 2j + 1)]$$

$$\Delta d_j \propto \Delta t^2 (2j - 1)$$

**Result:** For integers  $j = 1, 2, 3, \dots$ , the segment lengths scale as  $1, 3, 5, 7, \dots$ . This derived sequence corresponds directly to Galileo's experimental observations.

# Part 3: Scale Invariance

## A Pattern Within a Pattern

# Rescaling Time Intervals

A fundamental property of Galileo's sequence is its inherent scale invariance. The 1:3:5:7 ratio persists regardless of the temporal resolution used to observe the motion.

## Modifying the Observation Interval

Suppose we expand our basic observation block. Instead of evaluating displacements over intervals of duration  $\Delta t$ , we group the timeline into larger intervals, each possessing a duration of  $n\Delta t$ .

How does the summation of the underlying microscopic intervals correspond to the displacement measured across the macroscopic interval?

# The Summation Identity

This structural self-similarity can be expressed as a discrete algebraic identity concerning the summation of odd sequences.

$$\sum_{k=n(j-1)+1}^{nj} (2k - 1) = n^2(2j - 1)$$

## Physical Interpretation

Each relative length measured over the time interval  $n\Delta t$ , when scaled by  $n^2$ , equals the sum of the relative lengths measured over the  $n$  shorter intervals, each of duration  $\Delta t$ .

# Mathematical Consequences

Case  $j = 1$  establishes that the sum of the first  $n$  odd numbers is equal to  $n^2$ :

$$\sum_{k=1}^n (2k - 1) = n^2$$

Derivation of the sums for even numbers and all integers:

$$\sum_{k=1}^n (2k - 1) = n^2 \implies \sum_{k=1}^n 2k - n = n^2 \implies \sum_{k=1}^n 2k = n(n + 1)$$

$$\implies 2 \sum_{k=1}^n k = n(n + 1) \implies \sum_{k=1}^n k = \frac{n(n + 1)}{2}$$

# Conclusion

## Summary

- Galileo's Law of Odd Numbers is a direct consequence of the dimensionality of space, time, and acceleration.
- Dimensional analysis extracts the  $d \propto t^2$  relationship without requiring classical integration methods.
- The sequence exhibits mathematical scale invariance across varying temporal bounds.

## Note: Buckingham $\pi$ Theorem

Formalizing this approach: a system with  $m = 3$  variables ( $d, a, t$ ) and  $k = 2$  fundamental dimensions ( $L, T$ ) forms  $m - k = 1$  dimensionless group.

This single group is  $\pi_1 = \frac{d}{at^2}$ . Because it must equal a constant, the relationship  $d = Cat^2$  is inherently established.

# References & Acknowledgments

## References

- 1 A. De Angelis, *Galileo Galilei's "Two New Sciences": for Modern Readers* (History of Physics) (Springer, Cham, 2022), pp. 97-106.
- 2 L. Cooper, *Aristotle, Galileo, and the Tower of Pisa* (Cornell University Press, Ithaca, NY, 1935).
- 3 J. Fiala, "Analyzing Motion Diagrams: Uncovering Galileo's Law of Odd Numbers," *The Physics Teacher* (Accepted for publication).
- 4 M. Hall and J. Fiala, "Galileo's Law of Odd Numbers: The Odd Rhythm of Free Fall," *The Physics Teacher* (Accepted for publication).

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