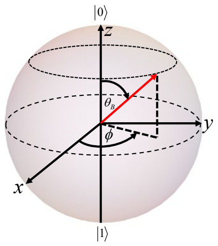

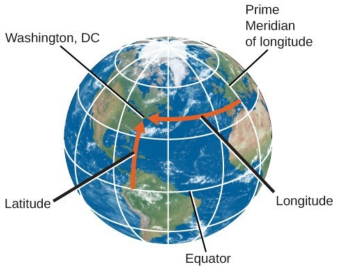

Designing Pedagogy for Quantum Entanglement

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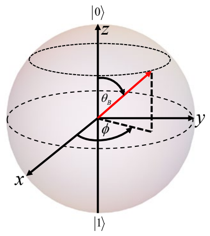
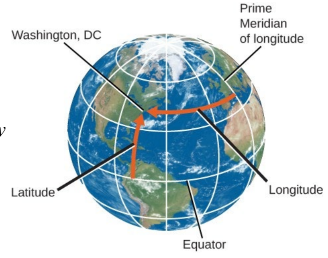
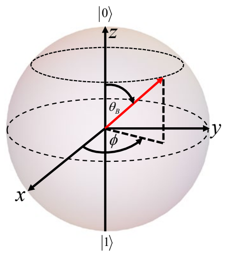
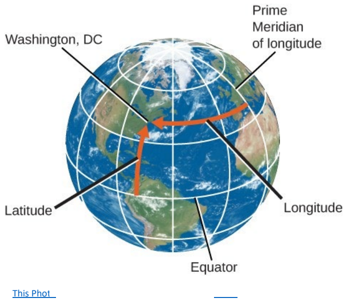




Introduction and Overview

- Entangled States – What's the big deal? Many important applications and conceptual puzzles.
- Need pedagogy that diminishes student misconceptions and misunderstandings
- Prepares students to understand QIS, QC, and Quantum Sensing Applications



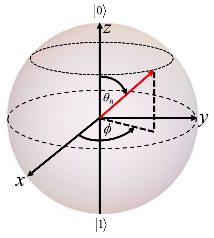


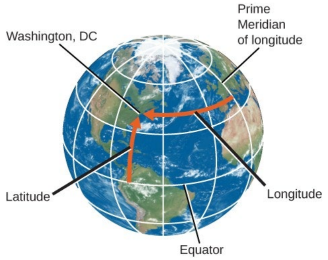
Multi-particle Quantum States

Example: two qubits

- Two spin-1/2 systems (not necessarily of the same type of particle)
- Four-dimensional Hilbert space = # of basis states
- Use basis states for spin along z axis (spin-up and spin-down).
- For the composite system basis states $\left|\uparrow_A\right\rangle \otimes \left|\uparrow_B\right\rangle = \left|\uparrow_A \uparrow_B\right\rangle$

Dirac notation





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Most general state of the composite system

$$|\psi\rangle = c_{\uparrow\uparrow}|\uparrow_A \uparrow_B\rangle + c_{\uparrow\downarrow}|\uparrow_A \downarrow_B\rangle + c_{\downarrow\uparrow}|\downarrow_A \uparrow_B\rangle + c_{\downarrow\downarrow}|\downarrow_A \downarrow_B\rangle$$

State coefficient

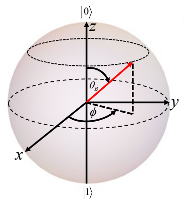
Product State – special case

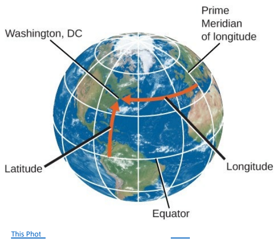
Independent qubits

$$|\psi_{\text{prod}}\rangle = \alpha_A |\Psi_A\rangle \alpha_B |\Psi_B\rangle$$

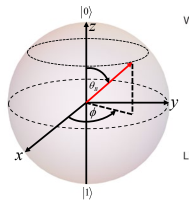
$|\psi\rangle$ can be written as a product state if and only if $c_{\uparrow\uparrow}c_{\downarrow\downarrow} = c_{\uparrow\downarrow}c_{\downarrow\uparrow}$

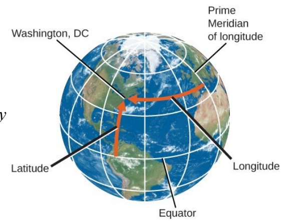
- Otherwise, the composite state is said to be entangled (non-separable).
- In that case, there are no quantum states describing the individual qubits.
- Correlations among measurements on the individual qubits.





- Entanglement means that there are correlations between measurement results.

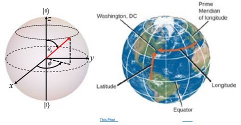




Traditional Approach to Entanglement

- Tensor product space
- Need to define operators in that tensor space
- Operate on both qubits
- Operate on only one qubit, leave the other alone
- Need to specify the state of the system after measurement on only one part
- Need correlation functions (expectation values of various operators) to extract entanglement features

Steep learning curve
Too many new ideas



Better Way: Focus on Probabilities

Entanglement correlation information

$$P(\uparrow_A \cap \uparrow_B) \qquad P(\uparrow_A) \qquad P(\uparrow_B | \uparrow_A)$$

joint total conditional

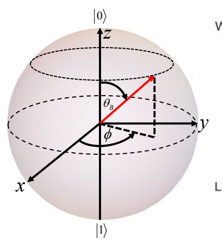
Joint Probability $P(\uparrow_A \cap \uparrow_B)$ Can be read off from composite system state vector

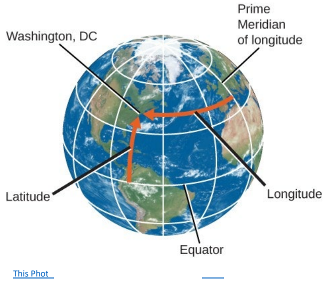
Total Probability $P(\uparrow_A) = P(\uparrow_A \cap \uparrow_B) + P(\uparrow_A \cap \downarrow_B)$

$$P(\uparrow_A \cap \uparrow_B) = P(\uparrow_A | \uparrow_B) P(\uparrow_B) = P(\uparrow_B | \uparrow_A) P(\uparrow_A)$$

Bayes's Rule (Theorem)

Conditional Probabilities





$$\left|\psi\right\rangle = c_{\uparrow\uparrow}\left|\uparrow_A\uparrow_B\right\rangle + c_{\uparrow\downarrow}\left|\uparrow_A\downarrow_B\right\rangle + c_{\downarrow\uparrow}\left|\downarrow_A\uparrow_B\right\rangle + c_{\downarrow\downarrow}\left|\downarrow_A\downarrow_B\right\rangle$$

joint Born Rule $P(\uparrow_A \cap \uparrow_B) = \left|\left\langle \uparrow_A\uparrow_B \middle| \psi\right\rangle\right|^2 = \left|c_{\uparrow\uparrow}\right|^2$

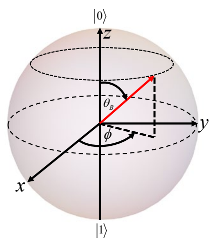
total $P(\uparrow_A) = \left|c_{\uparrow\uparrow}\right|^2 + \left|c_{\uparrow\downarrow}\right|^2$

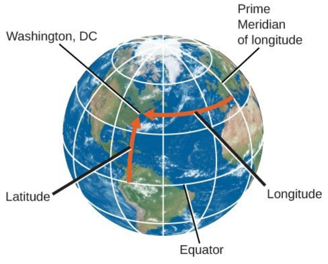
conditional

$$P(\uparrow_B|\uparrow_A) = \frac{P(\uparrow_A \cap \uparrow_B)}{P(\uparrow_A)} = \frac{\left|c_{\uparrow\uparrow}\right|^2}{\left|c_{\uparrow\uparrow}\right|^2 + \left|c_{\uparrow\downarrow}\right|^2}$$

$$P(\uparrow_B|\downarrow_A) = \frac{P(\downarrow_A \cap \uparrow_B)}{P(\downarrow_A)} = \frac{\left|c_{\downarrow\uparrow}\right|^2}{\left|c_{\downarrow\uparrow}\right|^2 + \left|c_{\downarrow\downarrow}\right|^2}$$

$\left|\psi\right\rangle$ can be written as a product state if and only $P\left(\uparrow_B \middle| \uparrow_A\right) = P\left(\uparrow_B \middle| \downarrow_A\right)$





$|\psi\rangle$ is an entangled state if and only if $P\left(\uparrow_B \middle| \uparrow_A\right) \neq P\left(\uparrow_B \middle| \downarrow_A\right)$

Three equivalent statements:

The two qubits are not statistically independent.

There are correlations among the measurements of A and B.

The composite system state is entangled.

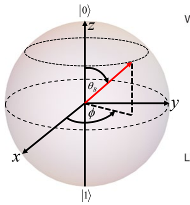
But Alice and Bob may be far apart. How do we interpret the correlated results?

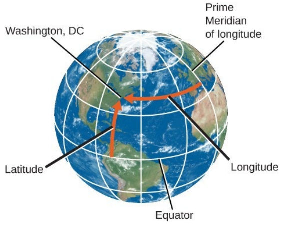
"Alice's observations affect the probability of Bob's measurement outcomes."

But correlation does not imply causation.

How does that correlation work? Simple explanations involving local hidden variables (i.e. a Newtonian physics approach) are ruled out by tests of Bell's Inequalities.

"Spooky action at a distance"?, A. Einstein





Summary

1. The interpretation of entangled states brings home the deep differences between quantum physics and classical (Newtonian) physics.

2. The composite system state vector contains all the information about measurement outcomes for the system: both collective and individual.

3. No need to invoke "collapse of the state vector."

4. Once the basics are in place, students are ready to learn about tensor products of operators, pure states, mixed states, states after measurements, correlation functions, etc.

Applications of Entangled States:

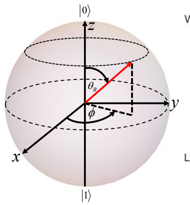
Quantum State Teleportation

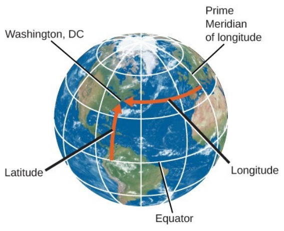
Bell's Theorem: (local hidden variables vs quantum mechanics)

Entanglement Swapping

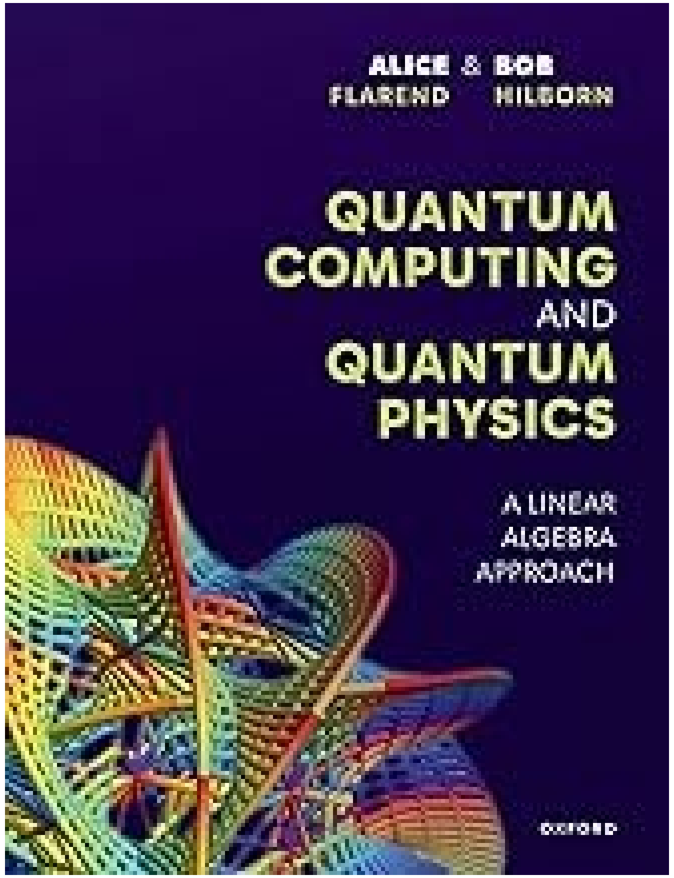
Cryptography, Superdense Coding

Enhanced quantum sensors: e.g. PET Scans, magnetic field sensors,…

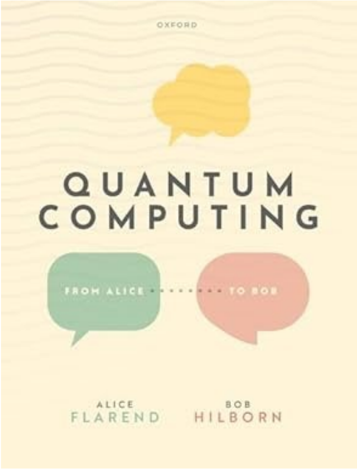




- Book: *Quantum Computing and Quantum Physics: A Linear Algebra Approach* (summer 2026)



Based on *Quantum Computing: From Alice to Bob,* Alice Flarend and Bob Hilborn (Oxford University Press, 2022)



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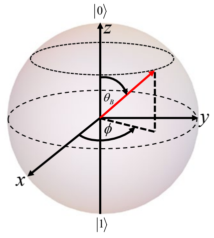

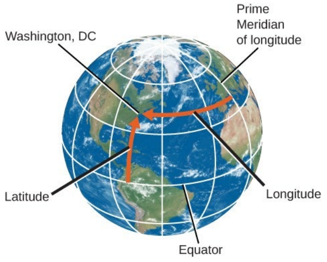

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Example

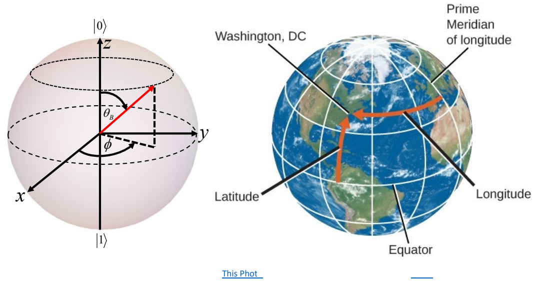
$$\left|\psi\right\rangle = \frac{1}{\sqrt{3}}\left(\left|\uparrow_A\uparrow_B\right\rangle + \left|\uparrow_A\downarrow_B\right\rangle + \left|\downarrow_A\downarrow_B\right\rangle\right)$$

Is this an entangled state?

$$P\left(\uparrow_A \cap \uparrow_B\right) = P\left(\uparrow_A \cap \downarrow_B\right) = P\left(\downarrow_A \cap \downarrow_B\right) = \frac{1}{3} \qquad P\left(\downarrow_A \cap \uparrow_B\right) = 0$$

$$P\left(\uparrow_A\right) = \frac{2}{3} \qquad P\left(\downarrow_A\right) = \frac{1}{3} \qquad P\left(\uparrow_B\right) = \frac{1}{3} \qquad P\left(\downarrow_B\right) = \frac{2}{3}$$

$$P\left(\uparrow_B \middle| \uparrow_A\right) = \frac{1}{2} \qquad P\left(\uparrow_B \middle| \downarrow_A\right) = 0$$



Correlation and Covariance

stochastic events E_A and E_B (numerical outcomes)

$$\operatorname{cov}(E_A, E_B) = \langle E_A E_B \rangle - \langle E_A \rangle \langle E_B \rangle$$

$$\operatorname{corr}(E_A, E_B) = \frac{\langle E_A E_B \rangle - \langle E_A \rangle \langle E_B \rangle}{\Delta E_A \Delta E_B}$$ normalized covariance

Quantum Computing and Quantum Physics: A Linear Algebra Approach provides a distinctive and accessible introduction to the rapidly growing fields of quantum information science and quantum computing.

While broadly accessible, the textbook does not dodge providing a solid conceptual and formal understanding of quantum states, superposition, and entanglement—the key ingredients in quantum computing. The authors dish up a hearty meal for the readers, disentangling many of the classic quantum algorithms that demonstrate how and when quantum computing has an advantage over classical computers.

The book includes pedagogical features such as *Try It* boxes, brief exercises that engage the readers and help them digest the many counterintuitive quantum information science and quantum computing concepts. Self-contained chapters on time dependence in quantum mechanics, magnetic resonance, density matrices, simple harmonic oscillator, quantum perturbation theory, and number operators provide a basis for further reading and work in quantum information science and quantum computing.

Aimed at undergraduate students with little or no background in physics, but with some familiarity with linear algebra, the book will also appeal to higher education faculty members and secondary school mathematics, physics, and computer science educators who want to learn about quantum computing and perhaps teach a course accessible to students with wide-ranging backgrounds.

ALICE FLAREND is a Physics Teacher at Bellwood-Antis High School and has two decades of experience providing teacher professional development in classical, nuclear, and quantum physics.

BOB HILBORN is Research Affiliate at the University of Maryland and has had many decades of experience doing research in atomic, molecular, and optical physics and teaching quantum mechanics to undergraduate students.

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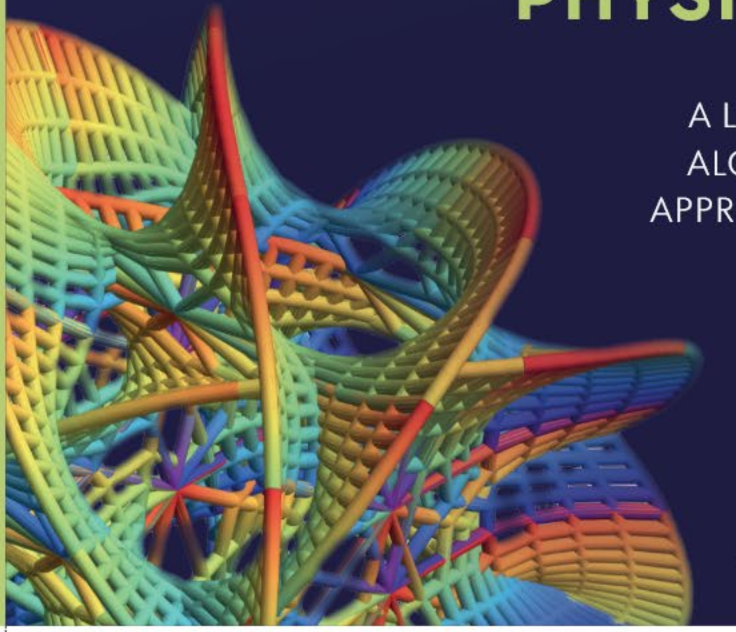

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